Ec2010a: Game Theory Section Notes

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"But I don't want to go among mad people," Alice remarked.
"Oh, you can't help that," said the Cat: "we're all mad here. I'm mad. You're mad."
"How do you know I'm mad?" said Alice.
"You must be," said the Cat, "or you wouldn't have come here."

— Alice in Wonderland, on mutual knowledge of irrationality

*These are an extended version of previous section notes by Kevin He and Jetlir Duraj. They contain additional exercises and material from older problem sets of Jerry Green, from the book *Game Theory* by Maschler, Solan, and Zamir and from the graduate book with the same title by Myerson. Please send comments and critiques to chang_liu@g.harvard.edu.

Ec2010a ▹ Game Theory Section 1: Welcome to Game Theory[1] —— 10/24/2021

(1) Course outline; **(2)** Normal form games; **(3)** Extensive form games; **(4)** Strategies in extensive form games; **(5)** Nash equilibrium and properties; **(6)** Optional: On the absent-minded driver

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1 Course Outline

1.1 *A taxonomy of games.* The second half of Ec2010a is organized around **several types of games**, paying particular attention to (i) relevant **solution concepts** in different settings, and (ii) some key **economic applications** belonging to these settings. To understand the course outline, it might be helpful to first introduce some binary classification schemes that give rise to these game types. Unfortunately, rigorous definitions of the following terminologies are not feasible without first laying down some background, so at this point we will instead appeal to hopefully familiar games to illustrate the classifications.

A game may have...

- Simultaneous moves (e.g. rock-paper-scissors) or sequential moves (e.g. checkers)
- Complete information (e.g. chess) or incomplete information (e.g. Stratego or Sanguosha)
- Chance moves (e.g. Backgammon) or no chance moves (e.g. Reversi)
- Finite horizon (e.g. tic-tac-toe) or infinite horizon (e.g. Gomoku on an infinite board)
- Zero-sum payoff structure (e.g. poker) or non-zero-sum payoff structure (e.g. the usual model of prisoner's dilemma)

1.2 *Course outline.* Roughly, the course can be divided into 4 units. Each unit is focused on one type of game, studying first its solution concepts then some important examples and applications. The relationship is summarized in Table 1.

Table 1: Solution concepts and economic applications.

	Complete information	Incomplete information
Simultaneous move	NE, rationalizability	BNE
Sequential move	SPE	PBE, SE, SSE, ...

	Complete information	Incomplete information
Simultaneous move	Nash implementation	Auctions
Sequential move	Repeated games	Signaling games

Game Type 1: **Simultaneous move** games with **complete information**

- Theory: Nash equilibrium (NE) and its extensions, rationalizability
- Application: Nash implementation

Game Type 2: **Simultaneous move** games with **incomplete information**

- Theory: Bayesian Nash equilibrium (BNE)
- Application: Auctions

[1] Figure 1 is from Haluk Ergin's game theory class at Berkeley. Figures for Example 13 and Example 14 are adapted from Maschler, Solan, and Zamir (2013).

GAME TYPE 3: **Sequential move** games with **complete information**

- Theory: Subgame perfect equilibrium (SPE)
- Application: Bargaining games, repeated games

GAME TYPE 4: **Sequential move** games with **incomplete information**

- Theory: Perfect Bayesian equilibrium (PBE), sequential equilibrium (SE), strategically stable equilibrium (SSE), etc.
- Application: Reputation, signaling games

1.3 *About sections.* Sections are optional. We will review lecture material and work out some additional examples. Please interrupt to ask questions. The use of the plural first-person pronoun "we" in these section notes does not indicate royal lineage or pregnancy – rather, it suggests the notes form a conversation between the writer and the audience.

2 Normal Form Games

2.1 *Interpreting the payoff matrix.* Here is the familiar **payoff matrix** representation of a two-player game.

Table 2: Game of assurance in normal form.

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$2, 2$

Player 1 (**P1**) chooses a row (*T*op or *B*ottom) while player 2 (**P2**) chooses a column (*L*eft or *R*ight). Each cell contains the payoffs to the two players when the corresponding pair of strategies is played. The first number in the cell is the payoff to P1 while the second number is the payoff to P2. (By the way, this game is sometimes called the "game of assurance".)

Two important things to keep in mind:

(1) In a normal form game, players choose their strategies **simultaneously**. That is, P2 cannot observe which row P1 picks when choosing his column.

(2) The terminology "payoff matrix" is slightly misleading. The numbers that appear in a payoff matrix are actually **Bernoulli utilities**, not monetary payoffs. To spell out this point in painstaking details: the set of possible outcomes of the game is a four-point set $X \equiv \{TL, TR, BL, BR\}$. Each player j has a preference $\succsim_j$ over $\Delta(X)$, the set of distributions on this set. Assume $\succsim_j$ satisfies the **von Neumann-Morgenstern (vNM)** axioms of independence and continuity. Then, running $\succsim_j$ through the vNM representation theorem, we find that $\succsim_j$ is represented by a utility function $U_j : \Delta(X) \to \mathbb{R}$ with the functional form $U_j(\boldsymbol{p}) = p_{TL} \cdot u_j(TL) + p_{TR} \cdot u_j(TR) + p_{BL} \cdot u_j(BL) + p_{BR} \cdot u_j(BR)$. We then enter $u_j(TL), u_j(TR), u_j(BL), u_j(BR)$ into the payoff matrix cells, which happen to be 1, 0, 0, 2.

In particular, in computing the expected utility of each player under a mixed strategy profile, we simply take a weighted average of the matrix entries – there is no need to apply a "utility function" to the entries before taking the average as they are already denominated in utils. Furthermore, it is important to remember that this kind of linearity does not imply risk-neutrality of the players, but is rather a property of the vNM representation.[2]

2.2 *General definition of a normal form game.* The payoff matrix representation of a game is convenient, but it is not sufficiently general. In particular, it seems unclear how we can represent games in which players have **infinitely** many possible strategies, such as a **Cournot duopoly**, in a finite payoff matrix. We therefore require a more general definition.

Definition 1 (Normal form game)**.** A **normal form game** $G = \left\langle N, (S_j)_{j \in N}, (u_j)_{j \in N} \right\rangle$ consists of:

[2] In fact, mixed strategies in game theory provided one of the motivations for von Neumann and Morgenstern's work on their representation theorem for preference over lotteries. von Neumann's theorem on the equality between maximin and minimax values in mixed strategies for zero-sum games assumed players choose the mixed strategy giving the highest expected value. But why should players choose between mixed strategies based on expected payoff rather than median payoff, mean payoff minus variance of payoff, or say the 4th moment of payoff? The vNM representation theorem rationalizes players maximizing expected payoff through a pair of conditions on their preference over lotteries.

1. A (finite) collection of **players** $N = \{1, 2, ..., n\}$.

2. A set of (pure) **strategies** S_j for each $j \in N$.

3. A (Bernoulli) **utility function** $u_j : S \to \mathbb{R}$ for each $j \in N$.

To interpret, the pure strategy set S_j is the set of actions that player j can take in the game. When each player chooses an action simultaneously from their own pure strategy set, we get a **strategy profile** $(s_1, s_2, ..., s_n) \in S$. Players derive payoffs by applying their respective utility functions to the strategy profile.

The payoff matrix representation of a game is a **specialization** of this definition. In a payoff matrix for 2 players, the elements of S_1 and S_2 are written as the names of the rows and columns, while the values of u_1 and u_2 at different members of $S_1 \times S_2$ are written in the cells. If $S_1 = \{s_1^A, s_1^B\}$ and $S_2 = \{s_2^A, s_2^B\}$, then the game $G = \langle\{1, 2\}, (S_1, S_2), (u_1, u_2)\rangle$ can be written in a payoff matrix:

	s_2^A	s_2^B
s_1^A	$u_1(s_1^A, s_2^A), u_2(s_1^A, s_2^A)$	$u_1(s_1^A, s_2^B), u_2(s_1^A, s_2^B)$
s_1^B	$u_1(s_1^B, s_2^A), u_2(s_1^B, s_2^A)$	$u_1(s_1^B, s_2^B), u_2(s_1^B, s_2^B)$

Conversely, the game of assurance can be converted into the standard definition by taking $N = \{1, 2\}$, $S_1 = \{T, B\}$, $S_2 = \{L, R\}$, $u_1(T, L) = 1$, $u_1(B, R) = 2$, $u_1(T, R) = u_1(B, L) = 0$, $u_2(T, L) = 1$, $u_2(B, R) = 2$, $u_2(T, R) = u_2(B, L) = 0$.

The general definition allows us to write down games with infinite strategy sets. In a **duopoly** setting where firms choose own production quantity, their choices are not taken from a finite set of possible quantities, but are in principle allowed to be any positive real number. So, consider a game with $S_1 = S_2 = [0, \infty)$,

$$u_1(s_1, s_2) = p(s_1 + s_2) \cdot s_1 - C(s_1),$$
$$u_2(s_1, s_2) = p(s_1 + s_2) \cdot s_2 - C(s_2),$$

where $p(\cdot)$ and $C(\cdot)$ are inverse demand function and cost function, respectively. Interpreting s_1 and s_2 as the quantity choices of firm 1 and firm 2, this is Cournot competition phrased as a normal form game.

2.3 *Recurring notations*. The following notations are common in game theory but usually go unexplained. Let $X_1, X_2, \ldots, X_n$ be a sequence of sets with typical elements $x_1 \in X_1$, $x_2 \in X_2$, ... Then:

- X_{-j} means $\prod_{1\leq k\leq n, k\neq j} X_k$ where $\prod$ denotes **Cartesian product**.
- X sometimes means $\prod_{j=1}^n X_j$.
- $x = (x_j)_{j=1}^n \in X$ refers to a vector[3] $(x_1, x_2, \ldots, x_n)$.
- x_{-j} is an element in X_{-j}, i.e., $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

To see an example of these notations, suppose we are studying a three player game

$$G = \langle\{1, 2, 3\}, (S_1, S_2, S_3), (u_1, u_2, u_3)\rangle .$$

Then s_{-2} refers to a vector containing strategies from player 1 and player 3, but not player 2. It is an element of $S_1 \times S_3$, also written as S_{-2}.

2.4 *Mixed strategies in normal form games.* A player who uses a mixed strategy in a game **intentionally introduces randomness** into her play. Instead of picking a deterministic action as in a pure strategy, a mixed strategy user tosses a coin to determine what action to play. Game theorists are interested in mixed strategies for at least two reasons: (i) mixed strategies correspond to **how humans play certain games**, such as rock-paper-scissors; (ii) the space of mixed strategies represents a **convexification** of the action set S_i and convexity is required for many existence results.

Henceforth, denote by $\Delta(A)$ the set of probability distributions over a set A.

Definition 2 (Mixed strategy in normal form)**.** Suppose $G = \left\langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\right\rangle$ is a normal form game where each S_j is finite.[4] Then a **mixed strategy for player** j, σ_j, is a probability distribution over S_j. That is, $\sigma_j \in \Delta(S_j)$.

[3] Sometimes also called a **profile**.

[4] We can also define mixed strategies when the set of actions S_j is infinite. However, we would need to first equip S_j with some σ-algebra, then define a mixed strategy as a probability measure on this σ-algebra.

Sometimes the mixed strategy that puts probability p_1 on action $s_1^{(1)}$ and probability $1 - p_1$ on action $s_1^{(2)}$ is written as $p_1 s_1^{(1)} \oplus (1 - p_1) s_1^{(2)}$. The "$\oplus$" notation (in lieu of "+") is especially useful when $s_1^{(1)}, s_1^{(2)}$ are numbers, as to avoid confusing the mixed strategy with an arithmetic expression.

Two remarks:

1. When two or more players play mixed strategies, their randomizations are assumed to be **independent**.
2. Technically, **pure strategies also count as mixed strategies** – they are simply degenerate distributions on the action set. The term **strictly mixed** is usually used for a mixed strategy that puts **strictly positive probability** on every action.

When a profile of mixed strategies σ is played, the assumption on independent mixing, together with payoff matrix entries being Bernoulli utilities in a vNM representation, imply that player j gets utility:

$$\sum_{(s_1, s_2, \ldots, s_n) \in S} \sigma_1(s_1) \cdots \sigma_n(s_n) u_j(s_1, s_2, \ldots, s_n).$$

We will abuse notation and write $u_j(\sigma_j, \sigma_{-j})$ for this utility, extending the domain of u_j into mixed strategies. We observe the following fact which turns out to be very useful.

Fact 3. *For any fixed σ_{-j}, the map $\sigma_j \mapsto u_j(\sigma_j, \sigma_{-j})$ is affine, in the sense that*

$$u_j(\sigma_j, \sigma_{-j}) = \sum_{s_j \in S_j} \sigma_j(s_j) u_j(s_j, \sigma_{-j}).$$

That is, the payoff to playing σ_j against opponents' mixed strategy profile σ_{-j} is some **weighted average** of the $|S_j|$ numbers $(u_j(s_j, \sigma_{-j}))_{s_j \in S_j}$, where the weights are given by the probabilities that σ_j assigns to these different actions.

3 Extensive Form Games

***3.1** Definition of an extensive form game.* The rich framework of extensive form games can incorporate sequential moves, incomplete and perhaps asymmetric information, randomization devices such as dice and coins, etc. It is one of the most powerful modeling tools of game theory, allowing researchers to formally study a wide range of economic interactions. Due to this richness, however, the general definition of an extensive form game is somewhat cumbersome. Roughly speaking, an extensive form game is a **tree** endowed with some **additional structures**. These additional structures formalize the **rules of the game**: the timing and order of play, the information of different players, randomization devices relevant to the game, outcomes and players' preferences over these outcomes, etc.

Definition 4 (Extensive form game)**.** A (finite-horizon) **extensive form game** Γ consists of:

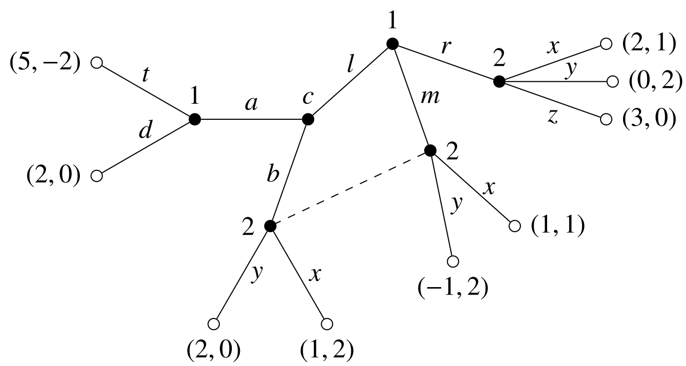
1. A (finite-depth) **tree** with vertices V and terminal vertices $Z \subseteq V$.
2. A (finite) collection of **players** $N = \{1, 2, ..., n\}$.
3. A **player function** $J : V \backslash Z \to N \cup \{c\}$. Denote $V_j = \{v : J(v) = j\}$ for each $j \in N \cup \{c\}$.
4. A set of available **moves** $M_{j,v}$ for each $j \in N$ and $v \in V_j$.
5. A **probability distribution** $f(\cdot|v)$ over v's children for each $v \in V_c$.
6. A (Bernoulli) **utility function** $u_j : Z \to \mathbb{R}$ for each $j \in N$.
7. An **information partition** $\mathcal{I}_j$ of V_j for each $j \in N$, whose elements are **information sets** $I_j \in \mathcal{I}_j$. It is required that $M_{j,v} = M_{j,v'}$ whenever $v, v' \in I_j$.

The game tree captures all possible states of the game. When players reach a terminal vertex $z \in Z$ of the game tree, the game ends and each player j receives utility $u_j(z)$. The player function J indicates who moves at each non-terminal vertex. The move might belong to an actual player $j \in N$, or to **chance**, "c". Note that V_j refers to the set of all vertices where player j has the move. If a player j moves at vertex v, she gets to pick an element from the set $M_{j,v}$ and play proceeds along the corresponding edge. If chance moves, then play proceeds along a random edge chosen according to $f(\cdot|v)$.

An information set I_j of player j refers to a set of vertices that player j **cannot distinguish between**.[5] It might be useful to imagine the players conducting the game in a lab, mediated by a computer. At each vertex $v \in V \backslash Z$, the computer finds the player $J(v)$ who has the move and informs her that the game has arrived at the information set $I_{J(v)} \ni v$. In the event that this $I_{J(v)}$ is a singleton, player $J(v)$ knows exactly her location in the game tree. Else, she knows only that she is at one of the vertices in $I_{J(v)}$, but she does not know for sure which one.[6] The requirement that two vertices in the same information set must have the same sets of moves is to prevent a player from gaining additional information by simply examining the set of moves available to her, which would defeat the idea that the player supposedly cannot distinguish between any of the vertices in the same information set. For convenience, we also write M_{I_j} for the common move set for all vertices $v \in I_j$.

There are two conventions for indicating an information set I_j in a game tree diagrams. Either all of the vertices in I_j are connected using dashed lines, or all of the vertices are encircled in an oval.

Example 5. Figure 1 illustrates all the pieces of the general definition of an extensive form game.



Chance move distributions: $f(a|(l)) = f(b|(l)) = 0.5$.
Information partitions: $\mathcal{I}_1 = \{\ \emptyset,\ (l, a)\ \}$, $\mathcal{I}_2 = \{\ (r),\ \{(l, b), (m)\}\ \}$.

Figure 1: An extensive form game with incomplete information and chance moves.

For convenience, let's name each vertex with the sequence of moves leading to it (and name the root as $\emptyset$). The set of players is $N = \{1, 2\}$. The player function $J(v)$ is shown on each $v \in V \backslash Z$ in Figure 1, while the payoff pair $(u_1(z), u_2(z))$ is shown on each $z \in Z$. The set of moves $M_{j,v}$ at vertex v is shown on the corresponding edges. Player 1 moves at two vertices, $V_1 = \{\ \emptyset,\ (l, a)\ \}$. Her information partition contains only singleton sets, meaning she always knows where in the game tree she is when called upon to move. Player 2 moves at three vertices, $V_2 = \{\ (r),\ (l, b),\ (m)\ \}$. However, player 2 cannot distinguish between (m) and (l, b), though he can distinguish (r) from the other two vertices. As such, his information partition contains two information sets, one containing just (r), the other containing the two vertices (m) and (l, b). As required by the definition, $M_{2,(m)} = M_{2,(l,b)} = \{x, y\}$, so that player 2 cannot figure out whether he is at (m) or (l, b) by looking at the set of available moves. ♦

The definition of extensive form games given above, allows us to formally characterize games of perfect information as well.

Definition 6 (Game of perfect information)**.** An extensive form game is called a **game of perfect information** if all the information sets of all players contain one node.

3.2 *Using information sets to convert a normal form game into extensive form.* Every finite normal form game $G = \left\langle N, (S_j)_{j=1}^n, (u_j)_{j=1}^n \right\rangle$ may be **converted** into an extensive form game of incomplete information with $1 + \sum_{m=1}^n \prod_{j=1}^m |S_j|$

[5] The use of an information partition to model a player's knowledge predates extensive form games with incomplete information. Such models usually specify a set of states of the world, Ω, then some partition $\mathcal{I}_j$ on Ω. When a state of the world $\omega \in \Omega$ realizes, player j is told the information set $I_j \in \mathcal{I}_j$ containing ω, but not the exact identity of ω. So then, **finer partitions** correspond to **better information**. To take an example, suppose 3 people are standing facing the same direction. An observer places a hat of one of two colors (say color 0 and color 1) on each of the 3 people. These 3 people cannot see their own hat color or the hat color of those standing behind them. Then the states of the world are $\Omega = \{000, 001, 010, 011, 100, 101, 110, 111\}$. The person in the front of the line has no information, so her information partition contains just one information set with all the states, $\mathcal{I}_1 = \{\ \{000, 001, 010, 011, 100, 101, 110, 111\}\ \}$. The second person in line sees only the hat color of the first person, so that $\mathcal{I}_2 = \{\ \{000, 010, 100, 110\},\ \{001, 011, 101, 111\}\ \}$. Finally, the last person sees the hats of persons 1 and 2, so that $\mathcal{I}_3 = \{\ \{000, 100\},\ \{001, 101\},\ \{010, 110\},\ \{011, 111\}\ \}$. In the context extensive form games, one might think of V_j as the relevant "states of the world" for j's decision-making and the fineness of her information partition $\mathcal{I}_j$ reflects the extent to which she can distinguish between these states.

[6] She might, however, be able to form a belief as to the likelihood of being at each vertex in $I_{J(v)}$, based on her knowledge of other players' strategies and the chance move distributions.

vertices. Construct a game tree with $n+1$ levels, so that all the vertices at level m belong to a single information set for player m, for $1 \le m \le n$. Level 1 contains the root. The root has $|S_1|$ children, corresponding to the actions in S_1. These children form the level 2 vertices. Each of these level 2 vertices has $|S_2|$ children, corresponding to the actions in S_2, and so forth. Each terminal vertex z in level $n+1$ corresponds to some action profile $(s_1^z, s_2^z, ..., s_n^z)$ in the normal form game G and is assigned utility $u_j(s_1^z, s_2^z, ..., s_n^z)$ for player j in the extensive form game. Figure 2 illustrates such a conversion using the game of assurance discussed earlier.

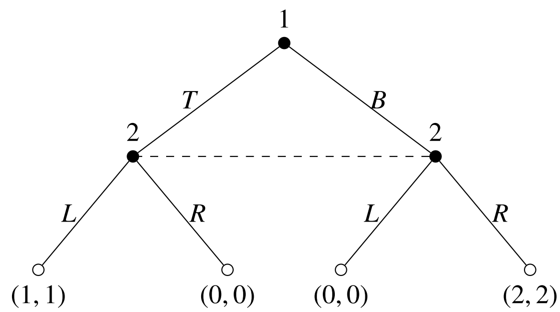


Figure 2: Game of assurance in extensive form.

4 Strategies in Extensive Form Games

***4.1** Pure strategy in extensive form games.* How would you write a program to play an extensive form game as player j? Whenever it is player j's turn, the program should take the **information set as an input** and return **one of the feasible moves as an output**. As the programmer does not *a priori* know the strategies that other players will use, the program must encode a **complete contingency plan** for playing the game so that it returns a legal move at every vertex of the game tree where j might be called upon to play. This motivates the definition of a pure strategy in an extensive form game.

Definition 7 (Pure strategy)**.** In an extensive form game, a **pure strategy** for player j is a function $s_j : \mathcal{I}_j \to \bigcup_{I_j \in \mathcal{I}_j} M_{I_j}$, so that $s_j(I_j) \in M_{I_j}$ for each $I_j \in \mathcal{I}_j$. Write S_j for the set of all pure strategies of player j.

That is, a pure strategy for player j returns a legal move at every information set of j.

Example 8. In Figure 1, one of the strategies of P1 is $s_1(\varnothing) = m$, $s_1(l, a) = d$. Even though playing m at the root means the vertex (l, a) will never be reached, P1's strategy must still specify what she **would have done** at (l, a). This is because some solution concepts we will study later in the course require us to examine parts of the game tree which are **unreached** when the game is played. Intuitively, this is necessary because the optimality of an action for a player at some information set may depend on what she/he and her/his opponents would have played on an information set which would be reached only if the player chooses differently than the strategy under consideration.

One of the strategies of P2 is $s_2(\{(l, b), (m)\}) = y$, $s_2(r) = z$. In every pure strategy P2 must play the same action at both (l, b) and (m), as pure strategies are functions of information sets, not individual vertices. In total, P1 has 6 different pure strategies in the game and P2 has 6 different pure strategies. ♦

***4.2** Two definitions of randomization.* There are at least two natural notions of "randomizing" in an extensive form game: (i) Player j could enumerate the set of all possible pure strategies, S_j, then choose an element of S_j at random; (ii) Player j could pick a randomization over M_{I_j} for each of her information sets $I_j \in \mathcal{I}_j$. These two notions of randomization lead to two different classes of strategies that incorporate stochastic elements:

Definition 9 (Mixed strategy)**.** A **mixed strategy** for player j is an element $\sigma_j \in \Delta(S_j)$.

Definition 10 (Behavioral strategy)**.** A **behavioral strategy** for player j is a collection of distributions $\{b_{I_j}\}_{I_j \in \mathcal{I}_j}$, where $b_{I_j} \in \Delta(M_{I_j})$.

Strictly speaking, mixed strategies and behavioral strategies form two **distinct classes of objects**. We may, however, talk about the equivalence between a mixed strategy and a behavioral strategy in the following way:

Definition 11. A mixed strategy σ_j and a behavioral strategy $\{b_{I_j}\}$ are **equivalent** if they generate the same distribution over terminal vertices regardless of the strategies used by opponents, which may be mixed or behavioral.

Note that in this definition for both the behavioral and the mixed case, opponents of j are assumed to play independently of each other.

Example 12. In Figure 1, a behavioral strategy for P1 is: $b^*_\varnothing(l) = 0.5$, $b^*_\varnothing(m) = 0$, $b^*_\varnothing(r) = 0.5$, $b^*_{(l,a)}(t) = 0.7$, $b^*_{(l,a)}(d) = 0.3$. That is, P1 decides that she will play m and r each with 50% probability at the root of the game. If she ever reaches the vertex (l, a), she will play t with 70% probability, d with 30% probability. But now, consider the following 4 pure strategies: $s_1^{(1)}(\varnothing) = l$, $s_1^{(1)}(l, a) = t$; $s_1^{(2)}(\varnothing) = l$, $s_1^{(2)}(l, a) = d$; $s_1^{(3)}(\varnothing) = r$, $s_1^{(3)}(l, a) = t$; $s_1^{(4)}(\varnothing) = r$, $s_1^{(4)}(l, a) = d$ and construct the mixed strategy σ^* so that $\sigma^*\left(s_1^{(1)}\right) = 0.35, \sigma^*\left(s_1^{(2)}\right) = 0.15, \sigma^*\left(s_1^{(3)}\right) = 0.35, \sigma^*\left(s_1^{(4)}\right) = 0.15$. Then the behavioral strategy b^* is equivalent to the mixed strategy σ^*. ♦

It is often "nicer" to work with behavioral strategies than mixed strategies, for at least two reasons. One, behavioral strategies are **easier to write down** and usually involve fewer parameters than mixed strategies. Two, it feels **more natural** for a player to randomize at each decision node than to choose a "grand plan" at the start of the game. In general, however, neither the set of mixed strategies nor the set of behavioral strategies is a "subset" of the other, as we now demonstrate.

Example 13 (A mixed strategy without an equivalent behavioral strategy)**.** Consider an **absent-minded city driver** who must make turns at two consecutive intersections. Upon encountering the second intersection, however, she does not remember whether she turned left (T) or right (B) at the first intersection. The mixed strategy σ_1 putting probability 50% on each of the two pure strategies T_1T_2 and B_1B_2 generates the outcome O_1 50% of the time and the outcome O_4 50% of the time. However, this outcome distribution cannot be obtained using any behavioral strategy. That is, if the driver chooses some probability of turning left at the first intersection and some probability of turning left at the second intersection, and furthermore these two randomizations are independent, then she can never generate the outcome distribution of 50% O_1, 50% O_4.

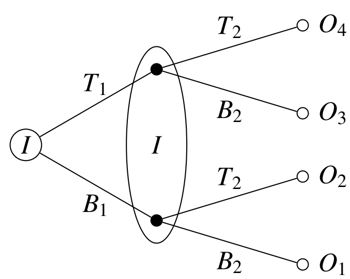


Figure 3: Absent-minded city driver.

♦

Example 14 (A behavioral strategy without an equivalent mixed strategy)**.** Consider an **absent-minded highway driver** who wants to take the second highway exit. Starting from the root of the tree, he wants to keep left (L) at the first highway exit but keep right (R) at the second highway exit. Upon encountering each highway exit, however, he does not remember if he has already encountered an exit before. The driver has only two pure strategies: always L or always R. It is easy to see no mixed strategy can ever achieve the outcome O_2. However, the behavioral strategy of taking L and R each with 50% probability each time he arrives at his information set gets the outcome O_2 with 25% probability.

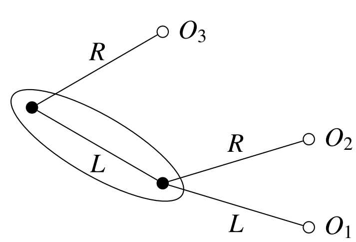


Figure 4: Absent-minded highway driver.

♦

These two examples are "pathological" in the sense that the drivers "forget" some information that they knew before. The city driver forgets what action she took at the previous information set. The highway driver forgets what information sets he has encountered. The definition of perfect recall rules out these two pathologies.

Definition 15 (Perfect recall)**.** An extensive form game has **perfect recall** if for each player j and information set I_j, whenever $v, v' \in I_j$, the two paths leading from the root to v and v' pass through the same sequence of information sets of player j, and player j takes the same actions at these information sets.

To put it another way, a game of perfect recall makes it impossible for a player who can remember all the information about the path of play she gathered in previous stages (i.e., never forgets anything) to find out in which node of any non-singleton information set she is located.

In the examples above: the city driver game fails perfect recall since taking two different actions from the root vertex lead to two vertices in the same information set. The highway driver game fails perfect recall since vertices x_1 and x_2 are in the same information set, yet the path from root to x_1 is empty while the path from root to x_2 passes through one information set.

Kuhn's theorem states that in a game with perfect recall, it is without loss to analyze only behavioral strategies. Its proof is beyond the scope of this course.

Theorem 16 (Kuhn, 1953)**.** *In a finite extensive game with perfect recall, (i) every mixed strategy has an equivalent behavioral strategy, and (ii) every behavioral strategy has an equivalent mixed strategy.*

5 Nash Equilibrium and Properties

5.1 *What does it mean to "solve" a game? A detour into combinatorial game theory*. Why are economists interested in Nash equilibrium, or solution concepts in general? As a slight aside, you may want to know that there actually exist two areas of research that go by the name of "game theory". The full names of these two areas are **combinatorial game theory** and **equilibrium game theory**. Despite the similarity in name, these two versions of game theory have quite different research agendas. The most salient difference is that combinatorial game theory studies well-known board games like chess where there exists (theoretically) a "winning strategy" for one player. Combinatorial game theorists aim to find these winning strategies, thereby solving the game. On the other hand, no "winning strategies" (usually called **dominant strategies** in our lingo) exist for most games studied by equilibrium game theorists[7]. In the game of assurance, for example, due to the simultaneous move condition, there is no one strategy that is optimal for P1 regardless of how P2 plays, in contrast to the existence of such optimal strategies in, say, tic-tac-toe.

If a game has a dominant strategy for one of the players, then it is straight-forward to predict its outcome under optimal play. The player with the dominant strategy will employ this strategy and the other player will do the best they can to minimize their losses. However, predicting outcome in a game without dominant strategies requires the analyst to make assumptions. These assumptions are usually called **equilibrium assumptions** and give equilibrium game theory its name. One of the most common equilibrium assumptions in normal form games with complete information is the **Nash equilibrium**, which we now study.

5.2 *Definition of Nash equilibrium.* A Nash equilibrium[8] is a strategy profile where no player can improve upon her own payoff through a **unilateral** deviation, taking as given the actions of others. This leads to the usual definition of pure and mixed strategy Nash equilibria.

Definition 17 (Pure strategy Nash equilibrium)**.** In a normal form game $G = \langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\rangle$, a **pure strategy Nash equilibrium** is a pure strategy profile s^* such that for every player j, $u_j(s_j^*, s_{-j}^*) \ge u_j(s_j', s_{-j}^*)$ for all $s_j' \in S_j$.

Definition 18 (Mixed strategy Nash equilibrium)**.** In a normal form game $G = \langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\rangle$, a **mixed strategy Nash equilibrium** is a mixed strategy profile σ^* such that for every player j, $u_j(\sigma_j^*, \sigma_{-j}^*) \ge u_j(s_j', \sigma_{-j}^*)$ for all $s_j' \in S_j$.

In the definition of a mixed Nash equilibrium, we required no profitable unilateral deviation to any **pure** strategy, s_i'. It would be equivalent to require no profitable unilateral deviation to any **mixed** strategy, due to the observation in Fact 3. If there is some profitable mixed strategy deviation σ_j' from a strategy profile $(\sigma_j^*, \sigma_{-j}^*)$, then it must be the case that for at least one $s_j' \in S_j$ with $\sigma_j'(s_j') > 0$, $u_j(s_j', \sigma_{-j}^*) > u_j(\sigma_j^*, \sigma_{-j}^*)$.

[7] The one-shot prisoner's dilemma is an exception here.

[8] John Nash called this equilibrium concept "equilibrium point" (Nash, 1950, 1951) but later researchers referred to it as "Nash equilibrium". We will see a similar situation later.

Example 19 (Game of assurance)**.** Consider the game of assurance,

	L	R
T	$1,1$	$0,0$
B	$0,0$	$2,2$

We readily verify that both (T, L) and (B, R) are pure strategy Nash equilibria. Note one of these two Nash equilibria Pareto dominates the other. In general, Nash equilibria **need not be Pareto efficient**. This is because the definition of NE only accounts for the absence of profitable **unilateral** deviations. Indeed, starting from the strategy profile (T, L), if P1 and P2 can contract on simultaneously changing their strategies, then they would both be better off. However, these sorts of simultaneous deviations by a "coalition" are not allowed.

But wait, there's more! Suppose P1 plays $\frac{2}{3}T \oplus \frac{1}{3}B$, and P2 plays $\frac{2}{3}L \oplus \frac{1}{3}R$. This strategy profile is a mixed NE. The reasoning is as follows. When P1 is playing $\frac{2}{3}T \oplus \frac{1}{3}B$, P2 gets an expected payoff of $\frac{2}{3}$ from playing L and an expected payoff of $\frac{2}{3}$ from playing R. Therefore, P2 has no profitable unilateral deviation because every strategy he could play, pure or mixed, would give the same payoff of $\frac{2}{3}$. Similarly, P2's mixed strategy $\frac{2}{3}L \oplus \frac{1}{3}R$ means P1 gets an expected payoff of $\frac{2}{3}$ whether she plays T or B, so P1 does not have a profitable deviation either. ♦

5.3 *Nash equilibrium as a fixed-point of the best response correspondence.* Nash equilibrium embodies the idea of stability. To make this point clear, it is useful to introduce an equivalent view of the Nash equilibrium through the lens of best response correspondences.

Definition 20 (Best response correspondence)**.** The **individual pure best response correspondence for player** j is $BR_j : S_{-j} \rightrightarrows S_j$[9] where

$$BR_j(s_{-j}) \equiv \underset{s'_j \in S_j}{\arg\max}\, u_j(s'_j, s_{-j}).$$

The **pure best response correspondence** is a vector of correspondences $BR : S \rightrightarrows S$ where

$$BR(s) \equiv (BR_j(s_{-j}))_{j \in N}.$$

Analogously, the **individual mixed best response correspondence for player** j is $\overline{BR}_j : \prod_{k \neq j} \Delta(S_k) \rightrightarrows \Delta(S_j)$ where

$$\overline{BR}_j(\sigma_{-j}) \equiv \underset{\sigma'_j \in \Delta(S_j)}{\arg\max}\, u_j(\sigma'_j, \sigma_{-j}).$$

The **mixed best response correspondence** is a vector of correspondences $\overline{BR} : \prod_{j \in N} \Delta(S_j) \rightrightarrows \prod_{j \in N} \Delta(S_j)$ where

$$\overline{BR}(\sigma) \equiv (\overline{BR}_j(\sigma_{-j}))_{j \in N}.$$

To interpret, the individual best response correspondences return the maximizer(s) of each player's utility function when opponents plays some known strategy profile. Depending on others' strategies, the player may have multiple maximizers, all yielding the same utility. As a result, we must allow the best responses to be **correspondences** rather than functions. Then, it is easy to see that:

Proposition 21. *A pure strategy profile is a pure strategy Nash equilibrium if and only if it is a fixed point of BR. A mixed strategy profile is a mixed strategy Nash equilibrium if and only if it is a fixed point of $\overline{BR}$.*

Fixed points of the best response correspondences reflect **stability** of NE strategy profiles, in the sense that even if player i knew what others were going to play, she still would not find it beneficial to change her actions. This rules out cases where a player plays in a certain way only because she held the **wrong expectations** about other players' strategies. We might expect such outcomes to arise initially when inexperienced players participate in the game, but we would also expect such outcomes to vanish as players **learn** to adjust their strategies to maximize their payoffs over time. That is to say, we expect non-NE strategy profiles to be unstable.

5.4 *Some properties of Nash equilibria.* Here are several important properties of NE. The first two are useful when computing NE:

Property 1: The indifference principle in mixed strategy Nash equilibria.

In Example 19, we saw that each action that one player plays with strictly positive probability yields the same expected payoff against the mixed strategy profile of the opponent. Turns out this is a general phenomenon.

[9] The notation $f : A \rightrightarrows B$ is equivalent to $f : A \to 2^B$.

Proposition 22. *If σ^* is a Nash equilibrium, then for s_j and s'_j such that $\sigma^*_j(s_j) > 0$ and $\sigma^*_j(s'_j) > 0$, we have $u_j(s_j, \sigma^*_{-j}) = u_j(s'_j, \sigma^*_{-j})$.*

Proof. It suffices to show that $u_j(s_j, \sigma^*_{-j}) = u_j(\sigma^*_j, \sigma^*_{-j})$ for any s_j such that $\sigma^*_j(s_j) > 0$. Suppose that, to the contrary, we may find $s_j \in S_j$ so that $\sigma^*_j(s_j) > 0$ but $u_j(s_j, \sigma^*_{-j}) \neq u_j(\sigma^*_j, \sigma^*_{-j})$.

1. If $u_j(s_j, \sigma^*_{-j}) > u_j(\sigma^*_j, \sigma^*_{-j})$, we contradict the optimality of σ^*_j in the maximization problem $\arg\max_{\sigma'_j \in \Delta(S_j)} u_j(\sigma'_j, \sigma_{-j})$, for we should have just picked $\hat{\sigma}_j = s_j$, the degenerate distribution on pure strategy s_j.
2. If $u_j(s_j, \sigma^*_{-j}) < u_j(\sigma^*_j, \sigma^*_{-j})$, we enumerate $S_j = \{s^{(1)}_j, ..., s^{(r)}_j\}$ and use the Fact 3 to expand:

$$u_j(\sigma^*_j, \sigma^*_{-j}) = \sum_{k=1}^{r} \sigma^*_j(s^{(k)}_j) \cdot u_j(s^{(k)}_j, \sigma^*_{-j}).$$

   The term $u_j(s_j, \sigma^*_{-j})$ appears in the summation on the right with a strictly positive weight, so if $u_j(s_j, \sigma^*_{-j}) < u_j(\sigma^*_j, \sigma^*_{-j})$ then there must exist another $s'_j \in S_j$ such that $u_j(s'_j, \sigma^*_{-j}) > u_j(\sigma^*_j, \sigma^*_{-j})$. But now we have again contradicted the fact that σ^*_j is a best mixed response to σ^*_{-j}.

This completes the proof. □

Property 2: In each Nash equilibrium, no player puts positive probability on a strictly dominated strategy.

Definition 23 (Strictly dominated)**.** A pure strategy $s_j \in S_j$ of player j is **strictly dominated** if there exists a (mixed) strategy $\sigma_j \in \Delta(S_j)$ such that for every $s_{-j} \in S_{-j}$,

$$u_j(\sigma_j, s_{-j}) > u_j(s_j, s_{-j}).$$

It turns out that recognizing strictly dominated strategies can simplify the analysis of the Nash equilibria of a game, since no player would ever employ them in a Nash equilibrium strategy.

Proposition 24. *If σ^* is a Nash equilibrium and s_j is strictly dominated, then $\sigma^*_j(s_j) = 0$.*

Proof. Suppose that s_j is strictly dominated by σ_j. This implies, in particular, that

$$u_j(\sigma_j, \sigma^*_{-j}) > u_j(s_j, \sigma^*_{-j}).$$

Since σ^*_j is a best response to σ^*_{-j} in the Nash equilibrium, it follows that

$$u_j(\sigma^*_j, \sigma^*_{-j}) \geq u_j(\sigma_j, \sigma^*_{-j}) > u_j(s_j, \sigma^*_{-j}).$$

The indifference principle then forces $\sigma^*_j(s_j)$ to be 0. □

As a corollary, **iterated elimination of strictly dominated strategies (IESDS)**[10] does not change the set of NE. In a game $G^{(1)}$, we can remove some or all of each player's strictly dominated strategies to arrive at a new game $G^{(2)}$, which will have the same set of NE as $G^{(1)}$. Furthermore, this procedure can be repeated, removing some of each player's strictly dominated strategies in $G^{(t)}$ to arrive at $G^{(t+1)}$. All of the games $G^{(1)}, G^{(2)}, G^{(3)}, ...$ will have the same set of NE, but computing NE of the later games is probably easier than computing NE of the original game $G^{(1)}$.

Next we turn to the mathematical properties of the set of Nash equilibria:

Property 3: The set of Nash equilibria of a finite game is **closed**, but in general **not convex**.

The set of Nash equilibria of a finite game is a subset of the product space of strategy profiles $\prod_{j \in N} \Delta(S_j)$.

- To see that it is a closed subset, consider a sequence of Nash equilibria of the game G, $\{\sigma^{(m)}\}$, that converges to a strategy profile σ^* as $m \to \infty$. We then have

$$u_j\left(\sigma^{(m)}_j, \sigma^{(m)}_{-j}\right) \geq u_j\left(\sigma_j, \sigma^{(m)}_{-j}\right), \quad \forall j \in N, \quad \forall \sigma_j \in \Delta(S_j).$$

  Note that the payoff functions are continuous in their arguments. We can pass to the limit to $m \to \infty$ to show

$$u_j\left(\sigma^*_j, \sigma^*_{-j}\right) \geq u_j\left(\sigma_j, \sigma^*_{-j}\right), \quad \forall j \in N, \quad \forall \sigma_j \in \Delta(S_j).$$

  This is just the definition of σ^* being a Nash equilibrium profile!

[10] Next section will focus more on IESDS and its properties. Some comments on the relation of the property of **never best response** to the property of being **strictly dominated**: These two concepts are equivalent for two-player games. For games with more than two players, they are in general not equivalent, under the usual assumption that opponents of a player cannot correlate their strategies.

- To give a counterexample to the set of Nash equilibria being convex, consider (again) the game of assurance.

	L	R
T	$1,1$	$0,0$
B	$0,0$	$2,2$

  (T, L) and (B, R) are Nash equilibria, but $\left(\frac{1}{2}T \oplus \frac{1}{2}B, \frac{1}{2}L \oplus \frac{1}{2}R\right)$ is not, as it violates the indifference principle. For instance, when P1 plays $\frac{1}{2}T \oplus \frac{1}{2}B$, P2 would deviate to playing R with probability one.

Finally, we turn to a particular class of games:

Definition 25 (Symmetric game)**.** $G = \left\langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\right\rangle$ is a **symmetric game** if

1. Each player has the same set of strategies: $S_j = S_k$ for all $j, k \in N$.
2. The payoff functions satisfy
$$u_{\pi(j)}(s_1, \ldots, s_n) = u_j(s_{\pi(1)}, \ldots, s_{\pi(n)})$$
   for any permutation π.

Examples of symmetric games are Bertrand duopoly or Cournot duopoly with identical costs, the prisoner's dilemma, etc.

Property 4: Every finite symmetric game has a symmetric mixed strategy Nash equilibrium (Nash, 1951).

One can adapt the proof of Nash's existence theorem to show that every finite symmetric game has a symmetric mixed strategy Nash equilibrium: an equilibrium σ^* satisfying $\sigma^*_j = \sigma^*_k$ for all $j, k \in N$.

This fact can come in handy when solving games. Moreover, the assumption of symmetric play is natural in the sense that players should be interchangeable in symmetric games (i.e., their identity doesn't matter for the game play).

6 Optional: On the Absent-Minded Driver

The two types of analyses presented here are intuitively nearer to the concepts of *Bayesian Nash equilibrium* and *correlated equilibrium* that we will cover in later parts of the lecture. They are based on Aumann, Hart, and Perry (1997).

Consider the absent-minded driver game we saw in the lecture:

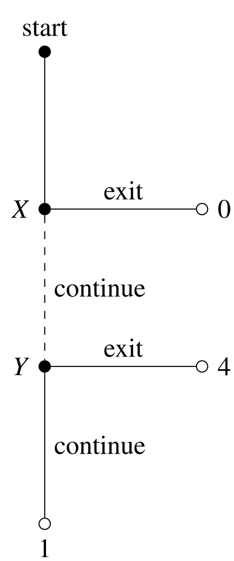


Figure 5: The absent-minded driver game.

We calculated in the lecture that the optimal behavioral strategy puts probability of $p = \frac{2}{3}$ on "continue". Here, in the first part, we consider another approach to the same problem, which takes beliefs of the driver about her location within the info set into account. In the second part, we show that with the help of simple correlating devices, it is possible to achieve an even higher payoff than with mixed or behavioral strategies.

A Bayesian perspective. For the following analysis we assume:

1. The driver makes a decision at *each* intersection through which he passes. Moreover, at any intersection, she can determine the action only *there* (she cannot determine the action at the other intersection).
2. Since she can't distinguish between intersections, whatever reasoning obtained at one intersection must be obtained also at the other, and she is aware of this.

This implies the following:

- The optimal decision is the same at both intersections; it is pinned down by the probability of choosing "continue" at each intersection. Call it p^*.
- Therefore, at each intersection, the driver believes that p^* is chosen at the other intersection.
- The driver has a belief over her location within her information set. At each intersection, the driver optimizes her decision given her beliefs. Therefore, choosing $p = p^*$ at the current intersection she is located, must be optimal given the belief that p^* is chosen at the other intersection. Moreover, her belief must be derived from the strategy she chooses.

By the *principle of indifference*[11] in Bayesian statistics, without any information about the strategy chosen, the probability of being at X will be $\frac{1}{2}$. Denote $\alpha(p^*)$ the belief the driver has about being at the intersection X, given her strategy of choosing p^* at the other intersection. The reasoning above and Bayes rule implies, that

$$\alpha(p^*) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}p^*}.$$

Given her beliefs about the behavior at the other node, the payoff of choosing p at the current node can be computed as

$$h(p, p^*) = \alpha(p^*)[(1-p)\cdot 0 + p(1-p^*)\cdot 4 + pp^*\cdot 1] + (1-\alpha(p^*))[(1-p)\cdot 4 + p\cdot 1] = \frac{(4-6p^*)p + 4p^*}{1+p^*}.$$

p must be chosen optimally, given the belief p^*. Moreover, p must be equal to p^*, since the agent doesn't distinguish between the nodes. That is, p^* must fulfill

$$p^* \in \arg\max_{p\in[0,1]} h(p, p^*).$$

The maximizing p for fixed p^* satisfies

$$p = \begin{cases} 0 & \text{if } p^* > \frac{2}{3} \\ \text{any value in } [0,1] & \text{if } p^* = \frac{2}{3} \\ 1 & \text{if } p^* < \frac{2}{3}. \end{cases}$$

This shows that the solution is unique and equal to $p^* = \frac{2}{3}$, the same as for the optimal behavioral strategy! Recall that the payoff of the optimal behavioral strategy is $\frac{4}{3}$.

The handkerchief solution. Assume the driver has a handkerchief in her pocket. Whenever she goes through an intersection, if there was no knot, she ties a knot in the handkerchief; if there was a knot, she unties it.

Assume that at the beginning, it is equally probable that the handkerchief had a knot or not. Assume that the driver is absent-minded in the sense that she cannot remember which was the case. Thus, at each one of the two intersections, the probability of having a knot in the handkerchief is $\frac{1}{2}$. Therefore, seeing a knot or not at each intersection does not reveal any information about the location of the intersection.

[11] The principle of indifference states that in the absence of any relevant evidence, agents should distribute their credence equally among all the possible outcomes under consideration.

Consider the following strategy for the driver: *exit if there is a knot, continue if there is not*. The payoff of this simple strategy is $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 = 2$: with probability $\frac{1}{2}$ the handkerchief had a knot in the first place, so that the driver exits and the payoff is 0; with probability $\frac{1}{2}$ the handkerchief had no knot so that the driver continues and ties a knot, and at the next node, seeing the knot the driver exits so that the payoff realized is 4.

Note that the path of play induced by this strategy can not be replicated using a behavioral strategy: the handkerchief allows the driver to avoid ever reaching the last node with payoff 1. Note also that $2 > \frac{4}{3}$, the payoff from the best behavioral strategy. The handkerchief has served as a coordination device between the forgetful selves in the different nodes and has achieved a higher payoff than the best behavioral strategy!

Ec2010a ▷ Game Theory Section 2: Nash Equilibrium and Correlated Equilibrium 10/31/2021

(1) Solving for Nash equilibria; **(2)** Correlated equilibrium; **(3)** Characterization of correlated equilibria

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1 Solving for Nash Equilibria

The following steps may be helpful in solving for Nash equilibria of two-player games.

1. Use **iterated elimination of strictly dominated strategies** to simplify the problem.
2. Find all the **pure strategy Nash equilibria** by considering all cells in the payoff matrix.
3. Look for **mixed strategy Nash equilibria** where one player is playing a pure strategy while the other is mixing.
4. Look for **mixed strategy Nash equilibria** where **both** players are mixing.

Example 26 (December 2013 Final Exam)**.** Find all Nash equilibria, pure and mixed, in the following payoff matrix.

	L	R	Y
T	$2, 2$	$-1, 2$	$0, 0$
B	$-1, -1$	$0, 1$	$1, -2$
X	$0, 0$	$-2, 1$	$0, 2$

Solution:
Step 1: Strategy X for P1 is strictly dominated by $\frac{1}{2}T \oplus \frac{1}{2}B$. Indeed, $u_1(X, L) = 0 < 0.5 = u_1\left(\frac{1}{2}T \oplus \frac{1}{2}B, L\right)$, $u_1(X, R) = -2 < -0.5 = u_1\left(\frac{1}{2}T \oplus \frac{1}{2}B, R\right)$, and $u_1(X, Y) = 0 < 0.5 = u_1\left(\frac{1}{2}T \oplus \frac{1}{2}B, Y\right)$. But having eliminated X for P1, strategy Y for P2 is strictly dominated by R: $u_2(T, Y) = 0 < 2 = u_2(T, R)$, $u_2(B, Y) = -2 < 1 = u_2(B, R)$. Hence we can restrict attention to the smaller, 2×2 game in the upper left corner.

Step 2: (T, L) is a pure Nash equilibrium as no player has a profitable unilateral deviation. (The deviation $L \to R$ does not **strictly** improve the payoff of P2, so it doesn't break the equilibrium.) At (T, R), P1 deviates $T \to B$, so it is not a pure strategy Nash equilibrium. At (B, L), P2 deviates $L \to R$. At (B, R), no player has a profitable unilateral deviation, so it is a pure strategy Nash equilibrium. In summary, the game has two pure strategy Nash equilibria: (T, L) and (B, R).

Step 3: Now we look for mixed Nash equilibria where one player is using a pure strategy while the other is using a strictly mixed strategy. As discussed before, if a player strictly mixes between two pure strategies, then she must be getting the same payoff from playing either of these two pure strategies.

Using this indifference principle, we quickly realize it cannot be the case that P2 is playing a pure strategy while P1 strictly mixes. Indeed, if P2 plays L then $u_1(T, L) > u_1(B, L)$. If P2 plays R then $u_1(B, R) > u_1(T, R)$.

Similarly, if P1 is playing B, then the indifference condition cannot be sustained for P2 since $u_2(R, B) > u_2(L, B)$.

Now suppose P1 plays T. Then $u_2(T, L) = u_2(T, R)$. This indifference condition ensures that any strictly mixed strategy of P2 $pL \oplus (1-p)R$ for $p \in (0, 1)$ is a mixed best response to P1's strategy. However, to ensure this is a mixed Nash equilibrium, we must **also make check** P1 does not have any profitable unilateral deviation. This requires:

$$u_1(T, pL \oplus (1-p)R) \geq u_1(B, pL \oplus (1-p)R).$$

That is to say,

$$2p + (-1) \cdot (1-p) \geq (-1) \cdot p + 0 \cdot (1-p) \quad \Leftrightarrow \quad p \geq \frac{1}{4}.$$

Therefore, $(T, pL \oplus (1-p)R)$ is a mixed Nash equilibrium where P2 strictly mixes when $p \in [\frac{1}{4}, 1)$.

Step 4: There are no mixed Nash equilibria where both players are strictly mixing. To see this, notice that if $\sigma_1^*(B) > 0$, then

$$u_2(\sigma_1^*, L) = 2 \cdot (1 - \sigma_1^*(B)) + (-1) \cdot (\sigma_1^*(B)) < 2 \cdot (1 - \sigma_1^*(B)) + (1) \cdot (\sigma_1^*(B)) = u_2(\sigma_1^*, R).$$

So it cannot be the case that P2 is also strictly mixing, since P2 is not indifferent between L and R.

To sum up, the game has two pure Nash equilibria, (T, L) and (B, R), as well as infinitely many mixed Nash equilibria, $(T, pL \oplus (1-p)R)$ for $p \in [\frac{1}{4}, 1)$. ♦

Sometimes, iterated elimination of strictly dominated strategy simplifies the game so much that the solution is immediate after this process. The following example illustrates.

Example 27 (Guess two-thirds the average, sometimes also called the beauty contest game[12]**).** Consider a game of 2 players $G^{(1)}$ where $S_1 = S_2 = [0, 100]$, $u_i(s_i, s_{-i}) = -\left(s_i - \frac{2}{3} \cdot \frac{s_i + s_{-i}}{2}\right)^2$. That is, each player wants to play an action as close to two-thirds the average of the two actions as possible.

We claim that for each player i, every action in $(50, 100]$ is strictly dominated by the action 50. To see this, for any opponent action $s_{-i} \in [0, 100]$, we have $\frac{2}{3} \cdot \frac{50 + s_{-i}}{2} \le 50$, so the guess 50 is already too high. At the same time, playing any $s_i > 50$ exacerbates the error relative to playing 50,

$$s_i - \frac{2}{3} \cdot \frac{s_i + s_{-i}}{2} > 50 - \frac{2}{3} \cdot \frac{50 + s_{-i}}{2} \ge 0.$$

Thus, $-\left(s_i - \frac{2}{3} \cdot \frac{s_i + s_{-i}}{2}\right)^2 < -\left(50 - \frac{2}{3} \cdot \frac{50 + s_{-i}}{2}\right)^2$ for all $s_i \in (50, 100]$ and we have the claimed strict dominance.

This means we may delete the set of actions $(50, 100]$ from each S_i to arrive at a new game $G^{(2)}$ where each player is restricted to using only $[0, 50]$. The game $G^{(2)}$ will have the same set of Nash equilibria as the original game. But the same logic may be applied again to show that in $G^{(2)}$, for each player, any action in $(25, 50]$ is strictly dominated by the action 25. We may continue in this way iteratively to arrive at a sequence of games $(G^{(k)})_{k \ge 1}$, so that in the game $G^{(k+1)}$, player i's action set is $\left[0, \left(\frac{1}{2}\right)^k \cdot 100\right]$. All of the games $G^{(1)}, G^{(2)}, G^{(3)}, ...$ have the same Nash equilibria. This means any NE of $G^{(1)}$ must involve each player playing an action in

$$\bigcap_{k=1}^{\infty} \left[0, \left(\frac{1}{2}\right)^k \cdot 100\right] = \{0\}.$$

Hence, $(0, 0)$ is the unique NE. ♦

Example 28. Consider the following three-player game, where the first player chooses rows, the second chooses columns and the third chooses the matrix.

X	L	R
T	$1, 1, 1$	$0, 1, 3$
B	$1, 3, 0$	$1, 0, 1$

Y	L	R
T	$3, 0, 1$	$1, 1, 0$
B	$0, 1, 1$	$0, 0, 0$

We claim that there is a unique Nash equilibrium and it is in pure strategies: (T, L, X).

Step 1: There is a unique pure strategy Nash equilibrium, (T, L, X).

To check that (T, L, X) is NE: if players 1 and 2 play (T, L), then player 3 is indifferent, so she might as well choose X. Note also, that (T, L) is NE of the two-player game created by taking matrix X and deleting all payoffs of player 3.

In this restricted game, there is also NE where players 1 and 2 play (B, L), but then player 3 would like to switch matrix to Y. If we restrict matrix Y to the payoffs of players 1 and 2 only, we see that there is only one pure strategy NE in the two-player induced game: (T, R), but then player 3 would like to X.

In all, there is only one pure strategy Nash equilibrium, (T, L, X).

Step 2: There are no Nash equilibria, where two players play pure strategies and the remaining player strictly mixes.

[12]The name "beauty contest game" comes from Keynes. He described the action of rational agents in a market using an analogy based on a fictional newspaper contest, in which entrants are asked to choose the six most attractive faces from a hundred photographs. Those who picked the most popular faces are then eligible for a prize. A naive strategy would be to choose the face that, in the opinion of the entrant, is the most handsome. A more sophisticated contest entrant, wishing to maximize the chances of winning a prize, would think about what the majority perception of attractive is, and then make a selection based on some inference from their knowledge of public perceptions. This can be carried one step further to take into account the fact that other entrants would each have their own opinion of what public perceptions are. Thus the strategy can be extended to the next order and the next and so on, at each level attempting to predict the eventual outcome of the process based on the reasoning of other rational agents. Here, we consider the more explicit scenario that helps to convey the notion of the contest as a convergence to Nash equilibrium, due to Ledoux (1981).

1. Player 3 strictly mixes. The indifference principle implies that players 1 and 2 play (T, L). But then player 2 would deviate to R, as she can guarantee payoff 1 even when Y is played (this happens with positive probability). Contradiction!

2. Player 2 strictly mixes. The indifference principle implies that players 1 and 3 play (T, X). But then player 1 would deviate to B, as she can guarantee payoff 1 even when R is played (this happens with positive probability). Contradiction!

3. Player 1 strictly mixes. The indifference principle implies that players 2 and 3 play (L, X). But then player 3 would deviate to Y, as she can guarantee payoff 1 even when B is played (this happens with positive probability). Contradiction!

Step 3: There are no Nash equilibria, where one player plays pure strategy and the remaining players strictly mix.

1. Player 1 plays a pure strategy. If T is played, player 3 is willing to strictly mix only if player 2 plays L with probability one; otherwise, she would choose X with probability one, contradiction! If B is played, player 2 would choose L with probability one, contradiction!

2. Player 2 plays a pure strategy. If L is played, player 1 is willing to strictly mix only if player 3 plays X with probability one; otherwise, she would choose T with probability one, contradiction! If R is played, player 3 would choose X with probability one, contradiction!

3. Player 3 plays a pure strategy. If X is played, player 1 is willing to strictly mix only if player 2 plays L with probability one; otherwise, she would choose B with probability one, contradiction! If Y is played, player 1 would choose T with probability one, contradiction!

Step 4: There are no Nash equilibria, where all players strictly mix.[13]

Let (p, q, r) be the probabilities with which, respectively, player 1 plays T, player 2 plays L and player 3 plays X.

The indifference condition for player 1 is

$$1 \cdot qr + 0 \cdot (1-q)r + 3q(1-r) + 1 \cdot (1-q)(1-r) = 1 \cdot r + 0 \cdot (1-r) \quad \Rightarrow \quad 3q(1-r) = (1-q)(2r-1).$$

Similarly, the indifference condition for player 2 and 3 can be written as, respectively,

$$3r(1-p) = (1-r)(2p-1) \quad \text{and} \quad 3p(1-q) = (1-p)(2q-1).$$

In particular, these three indifference conditions imply that $p, q, r > \frac{1}{2}$. Now multiply the three equations side by side. We get

$$27pqr = (2p-1)(2q-1)(2r-1).$$

RHS is smaller than 1, but LHS is greater than $\frac{27}{8}$, contradiction!

To sum up, there is a unique Nash equilibrium and it is in pure strategies: (T, L, X). ♦

2 Correlated Equilibrium

Let's begin with the definition of a correlated equilibrium in a normal form game.

Definition 29 (Correlated equilibrium)**.** In a normal form game $G = \left\langle N, (S_j)_{j \in N}, (u_j)_{j \in N} \right\rangle$, a **correlated equilibrium** (Ω, p, s^*) consists of:

1. A (finite) set of **signals** Ω_j for each $j \in N$. $\Omega \equiv \prod_{j \in N} \Omega_j$.

2. A (joint) **distribution** $p \in \Delta(\Omega)$, so that the marginal distributions $p(\omega_j) > 0$ for each $\omega_j \in \Omega_j$.

3. A (signal-dependent) **strategy** $s_j^* : \Omega_j \to S_j$ for each $j \in N$ such that for every $j \in N$, $\omega_j \in \Omega_j$, $s_j' \in S_j$,

$$\sum_{\omega_{-j} \in \Omega_{-j}} p(\omega_{-j}|\omega_j) u_j(s_j^*(\omega_j), s_{-j}^*(\omega_{-j})) \geq \sum_{\omega_{-j} \in \Omega_{-j}} p(\omega_{-j}|\omega_j) u_j(s_j', s_{-j}^*(\omega_{-j})).$$

[13] There are several ways to do this step, this is just one of them.

A correlated equilibrium envisions the following situation. At the start of the game, an n-dimensional vector of signals ω realizes according to the distribution p. Player j observes only the j-th dimension of the signal, ω_j, and plays an action $s_j^*(\omega_j)$ as a function of the signal she sees. Whereas a pure strategy Nash equilibrium has each player playing one action and requires that no player has a profitable unilateral deviation, in a correlated equilibrium each player may take **different actions depending on her signal**. Correlated equilibrium requires that no player can strictly improve her expected payoffs after seeing any of her signals. More precisely, seeing the signal ω_j leads her to have some belief over the signals that **others must have seen**, formalized by the conditional distribution $p(\cdot|\omega_j) \in \Delta(\Omega_{-j})$. Since she knows how these opponent signals translate into opponent actions through s_{-j}^*, she can compute the expected payoffs of taking different actions after seeing signal ω_j. She finds it optimal to play the action $s_j^*(\omega_j)$ instead of deviating to any other $s_j' \in S_i$ after seeing signal ω_j.

We make four remarks about correlated equilibria.

1. The signal space and its associated joint distribution, (Ω, p), are **not part of the game** G, but part of the equilibrium. That is, a correlated equilibrium constructs an **information structure** under which a particular outcome can arise.

2. There is no institution compelling player j to play the action $s_j^*(\omega_j)$, but j finds it optimal to do so after seeing the signal ω_j. It might be helpful to think of the **traffic lights** as an analogy for a correlated equilibrium. The light color that a player sees as she arrives at the intersection is her signal and imagine a world where there is no traffic police or cameras enforcing traffic rules. Each driver would nevertheless still find it optimal to stop when she sees a red light, because she infers that her seeing the red light signal must mean the driver on the intersecting street received the green light signal, and further the other driver is playing the strategy of going through the intersection if he sees a green light. Even though the red light (ω_j) merely **recommends** an action ($s_j^*(\omega_j)$), j finds it optimal to **obey this recommendation** given how others are acting on their own signals.

3. A Nash equilibrium is always a correlated equilibrium. Indeed, if σ^* is a Nash equilibrium in a normal form game $G = \left\langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\right\rangle$, construct signal spaces $\Omega_j = S_j$, define the distribution $p \in \Delta(\Omega)$ by

$$p(s_1, \ldots, s_n) = \sigma_1(s_1) \cdots \sigma_n(s_n),$$

   and consider the signal-dependent strategies $s_j^* : \Omega_j \to S_j$, $s_j^*(s_j) = s_j$. It is trivial to see that this gives a correlated equilibrium. In particular, correlated equilibria always exist.

4. The set of correlated equilibria of a finite normal form game is **convex**. Recall that this is not true for Nash equilibria. To see this intuitively, consider the following examples.

Example 30 (Game of assurance)**.** Consider the game of assurance,

	L	R
T	$1,1$	$0,0$
B	$0,0$	$2,2$

We have seen that (T, L) and (B, R) are Nash equilibria, but $\left(\frac{1}{2}T \oplus \frac{1}{2}B, \frac{1}{2}L \oplus \frac{1}{2}R\right)$ is not. Here is a correlated equilibrium where player 1 plays $\frac{1}{2}T \oplus \frac{1}{2}B$ and player 2 plays $\frac{1}{2}L \oplus \frac{1}{2}R$ "effectively": $\Omega_1 = \{t, b\}$, $\Omega_2 = \{l, r\}$, $p(t,l) = p(b,r) = 0.5$, $p(t,r) = p(b,l) = 0$, $s_1^*(t) = T$, $s_1^*(b) = B$, $s_2^*(l) = L$, $s_2^*(r) = R$. ♦

In this example, the signal structure is effectively a coordination device that picks the (T, L) Nash equilibrium 50% of the time, the (B, R) Nash equilibrium 50% of the time. Effectively, this correlated equilibrium can be thought of as flipping a coin, then instructing the players to play the (T, L) Nash equilibrium if heads up, and the (B, R) Nash equilibrium if tails up. We also refer to such coordination device as a **public randomization device**. This point can be made more general.

Example 31 (Public randomization device)**.** Fix any normal form game G and fix K of its pure Nash equilibria, $\overline{s}^{(1)}, ..., \overline{s}^{(K)}$. Then, for any probabilities $p_1, ..., p_K$ with $p_k > 0$, $\sum_{k=1}^K p_k = 1$, consider the signal space with $\Omega_j = \{1, ..., K\}$ for every $j \in N$, the joint distribution such that $p(k, ..., k) = p_k$ for each $1 \le k \le K$, and $p(\omega) = 0$ for any ω where not all n dimensions match, and the strategies $s_j^*(k) = \overline{s}_j^{(k)}$ for each $j \in N$, $1 \le k \le K$. Then (Ω, p, s^*) is a correlated equilibrium. Indeed, after seeing the signal k, each player knows that others must be playing their part of the k-th Nash equilibrium. As such, her recommended response $\overline{s}_j^{(k)}$ must be optimal. ♦

In general, fix K correlated equilibria $\{(\Omega^{(k)}, p^{(k)}, s^{*(k)})\}_{k=1}^{K}$ of a game and some strictly positive probability weights $(p_k)_{k=1}^{K}$, we can construct a new correlated equilibrium by first throwing a K-faced dice which falls on k with probability p_k, and instruct the players to play the k-th correlated equilibrium if face k realizes as outcome. The players will follow the instruction exactly because each $(\Omega^{(k)}, p^{(k)}, s^{*(k)})$ is a correlated equilibrium in the first place. This two-stage process gives a correlated equilibrium of the game, which is a mixture with weights $(p_k)_{k=1}^{K}$ of the original correlated equilibria.

Example 32 (Coordination game with an eavesdropper)**.** Three players Alice (P1), Bob (P2), and Eve (P3, the "eavesdropper") play a zero-sum game. Alice and Bob win only if they show up at the same location, and furthermore Eve is not there to spy on their conversation. The payoffs are given below. Alice chooses a row, Bob chooses a column, and Eve chooses a matrix.

L	L	R
L	$-1,-1,2$	$-1,-1,2$
R	$-1,-1,2$	$1,1,-2$

R	L	R
L	$1,1,-2$	$-1,-1,2$
R	$-1,-1,2$	$-1,-1,2$

The following is a correlated equilibrium. $\Omega_1 = \Omega_2 = \Omega_3 = \{l, r\}$, $p(l,l,l) = 0.25$, $p(l,l,r) = 0.25$, $p(r,r,l) = 0.25$, $p(r,r,r) = 0.25$, $s_i^*(l) = L$ and $s_i^*(r) = R$ for all $i \in \{1,2,3\}$. The information structure models a situation where Alice and Bob jointly observe some randomization device unseen by Eve[14] and use it to coordinate on either both playing L or both playing R. Eve's signals are uninformative of Alice and Bob's actions. Indeed, after seeing either $\omega_3 = l$ or $\omega_3 = r$, Eve thinks the chances are 50-50 that Alice and Bob are both playing L or both playing R, so she has no profitable deviation from the prescribed actions $s_3^*(l) = L$, $s_3^*(r) = R$. On the other hand, after seeing $\omega_1 = l$, Alice knows for sure that Bob is playing L while Eve has a 50-50 chance of playing L or R. Her payoff is maximized by playing the recommended $s_1^*(l) = L$. (Other deviations can be checked similarly.)

Eve's expected payoff in this correlated equilibrium is $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-2) = 0$. However, if Alice and Bob were to play **independent** mixed strategies, then Eve's best response leaves her with an **expected payoff of at least** 1. To see this, suppose Alice plays L with probability q_A and Bob plays L with probability q_B. If $q_A \cdot q_B \geq (1-q_A)\cdot(1-q_B)$, so that it is more likely that Alice and Bob coordinate on L than on R, Eve may play L to get an expected payoff of:

$$\underbrace{(-2)\cdot(1-q_A)\cdot(1-q_B)}_{\text{Alice and Bob meet without Eve}} + \underbrace{(2)\cdot[1-(1-q_A)\cdot(1-q_B)]}_{\text{otherwise}} \geq (-2)\cdot\frac{1}{4} + (2)\cdot\frac{3}{4} = 1,$$

where we used the fact that $q_A \cdot q_B \geq (1-q_A)\cdot(1-q_B) \Rightarrow q_A + q_B \geq 1 \Rightarrow (1-q_A)\cdot(1-q_B) \leq \frac{1}{4}$. On the other hand, if $q_A \cdot q_B \leq (1-q_A)\cdot(1-q_B)$, then Eve may play R to get an expected payoff of at least 1. ♦

3 Characterization of Correlated Equilibria

In general, it is impossible to characterize the set of correlated equilibria of a given game, due to the arbitrary choice of signal spaces. Yet every correlated equilibrium induces a probability distribution on the set of strategy profiles, and such distributions are often the main object of analyses when one applies the concept of correlated equilibrium. The formal definition of correlated equilibrium distributions is as follows:

Definition 33 (Correlated equilibrium distribution)**.** In a normal form game $G = \left\langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\right\rangle$, a **correlated equilibrium distribution** is a probability distribution $q \in \Delta(S)$ induced by a correlated equilibrium (Ω, p, s^*),

$$q = p \circ s^{*-1}.$$

That is,

$$q(s) = p(\{\omega \in \Omega : s^*(\omega) = s\}).$$

For instance, the correlated equilibrium in Example 30 induces the following correlated equilibrium distribution: $q(T,L) = q(B,R) = 0.5$, $q(T,R) = q(B,L) = 0$.

The set of correlated equilibrium distributions has a very convenient structure. It is a convex and compact subset of $\Delta(S)$, characterized by a set of linear inequalities.

[14] Perhaps an encrypted message.

Proposition 34. *In a (finite) normal form game* $G = \left\langle N, (S_j)_{j\in N}, (u_j)_{j\in N}\right\rangle$*, a probability distribution* $q \in \Delta(S)$ *is a correlated equilibrium distribution if and only if for each* s_j *with* $q(s_j) > 0$ *and for each* s'_j*,*

$$\sum_{s_{-j}\in S_{-j}} q(s_{-j}|s_j)u_j(s_j, s_{-j}) \geq \sum_{s_{-j}\in S_{-j}} q(s_{-j}|s_j)u_j(s'_j, s_{-j}). \tag{1}$$

The condition (1) is called the **obedience condition**. To see its logic, suppose that a disinterested moderator randomly selects a strategy profile s from the distribution q and recommends each player j to play s_j without giving any other information. Hearing the recommendation, player j comes to believe that the other players' strategies are distributed by $q(\cdot|s_j)$. The obedience condition states that she follows the recommendation.

Formally, this corresponds to the simple correlated equilibrium (Ω, p, s^*) with $\Omega = S$, $p = q$, $s^*(s) = s$. Hence, the obedience condition is a sufficient condition for a correlated equilibrium distribution. Conversely, in order to capture probability distributions induced by correlated equilibria with respect to arbitrary information structures, it suffices to consider this limited set of information structures. To see this, take any correlated equilibrium (Ω, p, s^*) and its induced correlated equilibrium distribution q on S. Now suppose that instead of letting j know that the realized state is ω_j, we only inform him that he needs to play $s^*_j(\omega_j)$ according to s^*_j. Since he did not have an incentive to deviate under any information (by definition of correlated equilibrium), by the sure-thing principle, he does not have an incentive to deviate now. Hence, the new model with limited information is also a correlated equilibrium. One crucial assumption that leads to this simplification is that u_j does not depend on ω_j.

Thanks to Proposition 34, the set of correlated equilibrium distributions is characterized by a finite set of linear inequalities:

$$\sum_{s_{-j}\in S_{-j}} \left(u_j(s_j, s_{-j}) - u_j(s'_j, s_{-j})\right) q(s_j, s_{-j}) \geq 0 \quad \forall j \in N, \quad \forall s_j, s'_j \in S_j.$$

Example 35 (Game of assurance)**.** We calculate in this example all correlated equilibrium distributions of the game of assurance,

	L	R
T	1, 1	0, 0
B	0, 0	2, 2

Denote a distribution on S by the following table:

	L	R
T	a	b
B	c	d

From the obedience conditions, this is a correlated equilibrium distribution if and only if

$$\begin{aligned}
&(1-0)a + (0-2)b \geq 0, && (\text{P1}, T)\\
&(0-1)c + (2-0)d \geq 0, && (\text{P1}, B)\\
&(1-0)a + (0-2)c \geq 0, && (\text{P2}, L)\\
&(0-1)b + (2-0)d \geq 0. && (\text{P2}, R)
\end{aligned}$$

The above conditions reduce to $a \geq 2b$, $a \geq 2c$, $2d \geq b$, $2d \geq c$. For all possible vectors of probabilities (a, b, c, d) that satisfies these conditions, there exists an associated correlated equilibrium.

Next consider the symmetric correlated equilibria, where $b = c$. Such symmetric distributions can be represented by pairs (a, b), with $a + 2b \leq 1$ and $d = 1 - a - 2b$. The above conditions further reduce to $b \leq a/2$ and $2a + 5b \leq 2$. The set of symmetric correlated equilibrium distributions is the shaded area in Figure 6.

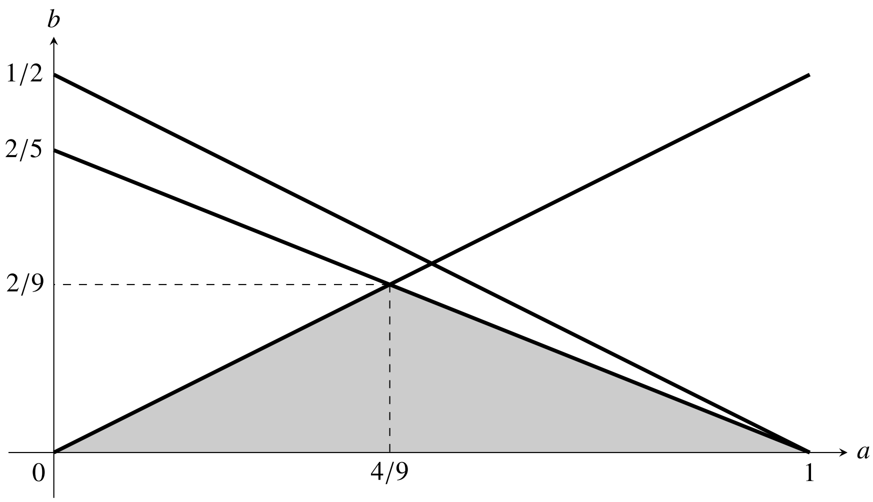

Figure 6: Symmetric correlated equilibria and Nash equilibria in the game of assurance.

Note that the Nash equilibria are also among the symmetric correlated equilibrium distributions where $(1, 0)$ is (T, L), $(0, 0)$ is (B, R) and $(4/9, 2/9)$ is the mixed strategy equilibrium $\left(\frac{2}{3}T \oplus \frac{1}{3}B, \frac{2}{3}L \oplus \frac{1}{3}R\right)$. ♦

In this example, the set of symmetric correlated equilibrium distributions is simply the convex hull of the Nash equilibrium distributions. Of course, there are also asymmetric correlated equilibria. In general, under broad conditions, the Nash equilibria are located on the boundary of the set of correlated equilibrium distributions.

Example 36. We calculate in this example all correlated equilibrium distributions of the game studied in the lecture,

	L	R
U	$5, 1$	$0, 0$
D	$4, 4$	$1, 5$

Denote a distribution on S by the following table:

	L	R
U	a	b
D	c	d

From the obedience conditions, this is a correlated equilibrium distribution if and only if

$$(5-4)a + (0-1)b \geq 0, \quad \text{(P1, } U\text{)}$$
$$(4-5)c + (1-0)d \geq 0, \quad \text{(P1, } D\text{)}$$
$$(1-0)a + (4-5)c \geq 0, \quad \text{(P2, } L\text{)}$$
$$(0-1)b + (5-4)d \geq 0. \quad \text{(P2, } R\text{)}$$

The above conditions reduce to $a \geq b$, $a \geq c$, $d \geq b$, $d \geq c$. For all possible vectors of probabilities (a, b, c, d) that satisfies these conditions, there exists an associated correlated equilibrium.

Note that $(1, 0, 0, 0)$ corresponds to the pure strategy Nash equilibrium (U, L), $(0, 0, 0, 1)$ corresponds to the pure strategy Nash equilibrium (D, R), and $(1/4, 1/4, 1/4, 1/4)$ corresponds to the mixed strategy Nash equilibrium $\left(\frac{1}{2}U \oplus \frac{1}{2}D, \frac{1}{2}L \oplus \frac{1}{2}R\right)$. Moreover, $(1/3, 0, 1/3, 1/3)$ corresponds to the correlated equilibrium constructed in the lecture. From $a \geq c$, $d \geq c$, and $a + c + d \leq 1$, we can conclude that $c \leq 1/3$. Hence, this correlated equilibrium is the one with the highest probability that the strategy profile (D, L) is played. One can proceed to show that this correlated equilibrium maximizes the sum of the players' expected payoffs. ♦

(1) Rationalizability; **(2)** Mechanism design and Nash implementation

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"But I don't want to go among mad people," Alice remarked.
"Oh, you can't help that," said the Cat: "we're all mad here. I'm mad. You're mad."
"How do you know I'm mad?" said Alice.
"You must be," said the Cat, "or you wouldn't have come here."

— Alice in Wonderland, on common knowledge of irrationality

1 Rationalizability

1.1 *Two algorithms.* Consider a normal form game G. Here we review the two algorithms of iterative strategy elimination studied in lecture.

Algorithm 37 (Iterated elimination of strictly dominated strategies, "**IESDS**")**.**

1. ***Initialize:*** $\hat{S}_i^{(0)} := S_i$ *for each i.*
2. ***For*** $t \geq 0$***:***
$$\hat{S}_i^{(t+1)} := \left\{ s_i \in \hat{S}_i^{(t)} : \nexists \sigma_i \in \Delta\left(\hat{S}_i^{(t)}\right) \text{ s.t. } \forall s_{-i} \in \hat{S}_{-i}^{(t)}, u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \right\}.$$
3. ***Output:***
$$\hat{S}_i^{\infty} := \bigcap_{t \geq 0} \hat{S}_i^{(t)}.$$

The idea behind IESDS is that if some mixed strategy σ_i yields **strictly** more payoff than the action s_i regardless of what other players do, then i will never use the action s_i. The "**iterated**" part comes from requiring that (i) the dominating mixed strategy must be supported on i's actions that **survived the previous rounds** of eliminations; (ii) the conjecture of what other players might do must be taken from their strategies that **survived the previous rounds** of eliminations.

Algorithm 38 (Iterated elimination of never best responses, "**IENBR**")**.**

1. ***Initialize:*** $\tilde{S}_i^{(0)} := S_i$ *for each i.*
2. ***For*** $t \geq 0$***:***
$$\tilde{S}_i^{(t+1)} := \left\{ s_i \in \tilde{S}_i^{(t)} : \exists \sigma_{-i} \in \Delta\left(\tilde{S}_{-i}^{(t)}\right) \text{ s.t. } \forall s_i' \in \tilde{S}_i^{(t)}, u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \right\}.$$
3. ***Output:***
$$\tilde{S}_i^{\infty} := \bigcap_{t \geq 0} \tilde{S}_i^{(t)}.$$

It is important to note that $\Delta\left(\tilde{S}_{-i}^{(t)}\right) \neq \prod_{k \neq i} \Delta\left(\tilde{S}_k^{(t)}\right)$ when $n \geq 3$.[15] The left-hand-side is the set of **correlated strategies** of players other than i, i.e., the set of all joint distributions on $\tilde{S}_{-i}^{(t)}$. Such a correlated mixed strategy might be generated, for example, using a signal-space kind of setup as in correlated equilibrium. The elimination of never best responses can be viewed as asking each action of player i to "justify its existence" by naming a correlated mixed strategy[16] of opponents for which it is a best response. The "**iterated**" part comes from requiring that this conjecture of correlated opponents' strategy have support in their strategies that **survived the previous rounds** of eliminations.

[15] When there are only two players, equality does hold: $\Delta\left(\tilde{S}_{-i}^{(t)}\right) = \prod_{k \neq i} \Delta\left(\tilde{S}_k^{(t)}\right)$. This is because $-i$ refers to exactly 1 player, not a group of players, so we do not get anything new by allowing $-i$ to "correlate amongst themselves". As such, we did not have to worry about correlated vs. independent opponent strategies when we computed rationalizable strategy profiles for a two-player game in lecture.

[16] This correlated opponents' strategy might reflect i's belief that opponents are colluding and coordinating their actions, or it could reflect correlation in i's subjective uncertainty about what two of her opponents might do.

Another view on these two algorithms is that they make progressively **sharper predictions** about the game's outcome by making more and more levels of **rationality assumptions**. A "rational" player i is someone who maximizes the utility function u_i as given in the normal form game G. Rational players are contrasted against the so-called "**crazies**" present in some models of reputation, who are mad in the sense of either maximizing a different utility function than normal players, or in not choosing actions based on utility maximization at all. From the analysts' perspective, knowing that every player is rational allows us to predict that only actions in $\hat{S}_i^{(1)}$ (equivalently, $\tilde{S}_i^{(1)}$) will be played by i, since playing any other action is incompatible with maximizing u_i. But we cannot make more progress unless we are **also willing to assume** what i knows about j's rationality. If i is rational but i thinks that j might be mad, in particular that j might take an action in $S_j \backslash \hat{S}_j^{(1)}$, then the step for constructing $\hat{S}_i^{(2)}$ for i does not make sense. As it is written in Algorithm 37, we should eliminate any action of i that does strictly worse than a fixed mixed strategy against all action profiles taken from $\hat{S}_{-i}^{(1)}$, which in particular assumes that j must be playing something in $\hat{S}_j^{(1)}$. In general, the t-th step for each of Algorithm 37 and Algorithm 38 rests upon assumptions of the form "i knows that j knows that ... that k is rational" with length t.

1.2 *Equivalence of the two algorithms.* In fact, Algorithm 37 and Algorithm 38 are the equivalent, as we now demonstrate.[17]

Proposition 39. $\hat{S}_i^{(t)} = \tilde{S}_i^{(t)}$ *for each* $i \in N$ *and* $t \geq 0$. *In particular,* $\hat{S}_i^{\infty} = \tilde{S}_i^{\infty}$.

In view of this result, we call $\tilde{S}_i^{\infty}$ the **rationalizable strategies** of player i, but note that it can be computed through either IENBR or IESDS.

Proof. Do induction on t. When $t = 0$, $\hat{S}_i^{(0)} = \tilde{S}_i^{(0)} = S_i$ by definition. Suppose for each $i \in N$, $\hat{S}_i^{(t)} = \tilde{S}_i^{(t)}$.

To establish that $\tilde{S}_i^{(t+1)} \subseteq \hat{S}_i^{(t+1)}$, take some $s_i^* \in \tilde{S}_i^{(t+1)}$. By definition of IENBR, there is some $\sigma_{-i} \in \Delta\left(\tilde{S}_{-i}^{(t)}\right)$ s.t.

$$u_i\left(s_i^*, \sigma_{-i}\right) \geq u_i(s_i', \sigma_{-i}), \quad \forall s_i' \in \tilde{S}_i^{(t)}.$$

The inductive hypothesis lets us replaces all tildes with hats, so that there is some $\sigma_{-i} \in \Delta\left(\hat{S}_{-i}^{(t)}\right)$ s.t.

$$u_i\left(s_i^*, \sigma_{-i}\right) \geq u_i(s_i', \sigma_{-i}), \quad \forall s_i' \in \hat{S}_i^{(t)}.$$

If s_i^* were strictly dominated by some $\hat{\sigma}_i \in \Delta\left(\hat{S}_i^{(t)}\right)$, then $u_i(s_i^*, \sigma_{-i}) < u_i(\hat{\sigma}_i, \sigma_{-i})$, because the same strict inequality holds at every s_{-i} in the support of σ_{-i}. By Fact 3, there exists some $\hat{s}_i \in \hat{S}_i^{(t)}$ with $\hat{\sigma}_i(\hat{s}_i) > 0$ so that $u_i(s_i^*, \sigma_{-i}) < u_i(\hat{s}_i, \sigma_{-i})$, contradicting s_i^* being a best response to σ_{-i}.

Conversely, for the reverse inclusion $\tilde{S}_i^{(t+1)} \supseteq \hat{S}_i^{(t+1)}$, suppose $s_i^* \in \hat{S}_i^{(t+1)}$. Combining definition of IESDS and the inductive hypothesis shows that for each $\sigma_i \in \Delta\left(\tilde{S}_i^{(t)}\right)$, there corresponds some $s_{-i} \in \tilde{S}_{-i}^{(t)}$ so that $u_i\left(s_i^*, s_{-i}\right) \geq u_i(\sigma_i, s_{-i})$ (otherwise, s_i^* is strictly dominated by σ_i). Now enumerate $\tilde{S}_{-i}^{(t)} = \left\{s_{-i}^{(1)}, ..., s_{-i}^{(d)}\right\}$ and hence construct the following subset of of $\mathbb{R}^d$:

$$V \equiv \left\{v \in \mathbb{R}^d : \exists \sigma_i \in \Delta\left(\tilde{S}_i^{(t)}\right) \text{ s.t. } v_k \leq u_i\left(\sigma_i, s_{-i}^{(k)}\right),\ \forall 1 \leq k \leq d\right\}.$$

That is, every $\sigma_i \in \Delta\left(\tilde{S}_i^{(t)}\right)$ gives rise to a point $\left(u_i(\sigma_i, s_{-i}^{(1)}), \ldots, u_i(\sigma_i, s_{-i}^{(d)})\right) \in \mathbb{R}^d$ and V is the region to the "lower-left" of this collection of points. We can verify that V is convex and non-empty. Now consider the point

$$w = \left(u_i(s_i^*, s_{-i}^{(1)}), \ldots, u_i(s_i^*, s_{-i}^{(d)})\right) \in \mathbb{R}^d.$$

We must have $w \notin \text{int}(V)$, where $\text{int}(V)$ is the interior of V. As such, **separating hyperplane theorem** implies there is some $q \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ with $q \cdot w \geq q \cdot v$ for all $v \in V$. Since V includes points with arbitrarily large negative numbers in each coordinate, we must in fact have $q \in \mathbb{R}_+^d \backslash \{\mathbf{0}\}$, i.e., q cannot have a negative coordinate. So then, q may be normalized so that its coordinates add up to 1, and thus it can be viewed as some correlated strategy $\sigma_{-i}^* \in \Delta\left(\tilde{S}_{-i}^{(t)}\right)$. This strategy

[17] Note that the finiteness of the strategy space is also important for this equivalence result. To see a counterexample, consider the following two-player game with infinite strategies: $S_1 = \{\textbf{swap}, 1, 2, 3, \ldots\}$, $S_2 = \{1, 2, 3, \ldots\}$,

$$u_1(s_1, s_2) = \begin{cases} s_2 & \text{if } s_1 = \textbf{swap}, \\ s_1 & \text{otherwise.} \end{cases}$$

In other words, player 1 can secure as payoff any positive integer she picks, but she can also swap for the integer that player 2 picks. In this game, the strategy **swap** for player 1 is not strictly dominated by any mixed strategy (with finite mean): **swap** works better when player 2 chooses an integer larger than the mean of that mixed strategy. However, swap is never a best response to any player 2's mixed strategy (with finite mean): **swap** is worse than player 1's choice of any integer larger than the mean of that mixed strategy.

has the property that $u_i\left(s_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right)$ for all $\sigma_i \in \Delta\left(\tilde{S}_i^{(t)}\right)$, showing that in particular s_i^* is a best response to σ_i^* amongst $\tilde{S}_i^{(t)}$, hence $s_i^* \in \tilde{S}_i^{(t+1)}$. This establishes the reverse inclusion $\hat{S}_i^{(t+1)} \subseteq \tilde{S}_i^{(t+1)}$ and completes the inductive step. □

1.3 *Rationalizability and equilibrium concepts*. In some sense, the collection of rationalizable strategies includes the collection of correlated equilibrium strategies. To be more precise,

Proposition 40. *If (Ω, p, s^*) is a correlated equilibrium, then $s_i^*(\omega_i) \in \tilde{S}_i^\infty$ for every $i \in N$ and $\omega_i \in \Omega_i$.*

Proof. We show for any player i and any s_i such that $s_i \in s_i^*(\Omega_i)$ (the image of mapping s_i^*), $s_k \in \tilde{S}_i^{(t)}$ for every t. This statement is clearly true when $t = 0$. Suppose this statement is true for $t = T$. Then, for each player i and each signal $\omega_i \in \Omega_i$, consider the correlated opponent strategy σ_{-i}^* constructed by

$$\sigma_{-i}^*(s_{-i}) \equiv p\left(\{\omega_{-i} \in \Omega_{-i} : s_{-i}^*(\omega_{-i}) = s_{-i}\} \,|\omega_i\right).$$

By definition of correlated equilibria, $s_i^*(\omega_i)$ best responds to σ_{-i}^*. Furthermore, $\sigma_{-i}^* \in \Delta\left(\tilde{S}_{-i}^{(T)}\right)$ by inductive hypothesis. Therefore, $\hat{s}_i \in \tilde{S}_i^{(T+1)}$, completing the inductive step. □

Therefore, we see that correlated equilibria (and in particular, Nash equilibria) **embed the assumption of common knowledge of rationality**: not only is Alice rational, but also Alice knows Bob is rational, and Alice knows that Bob knows Alice is rational, etc.

1.4 *Nested solution concepts*. Here we summarize the inclusion relationships between several solution concepts. For a normal form game G,

$$\text{Rat}(G) \supseteq \text{CE}(G) \supseteq \text{NE}(G).$$

2 Mechanism Design and Nash Implementation

2.1 *Mechanism design as a decentralized solution to the information problem.*

Definition 41 (Mechanism design problem)**.** A **mechanism design problem** consists of the following:

1. A (finite) collection of **players** $N = \{1, ..., n\}$.
2. A set of **states of the world** Θ.
3. A set of **outcomes** A.
4. A **state-dependent utility** $u_i : A \times \Theta \to \mathbb{R}$ for each player $i \in N$.
5. A **social choice rule** $f : \Theta \rightrightarrows A$.

Every mechanism design problem presents an information problem. Consider a designer who is **omnipotent** (all-powerful) but **not omniscient** (all-knowing). It can choose any outcome $a \in A$. However, the outcome it wants to pick depends on the state of the world. When the state of the world is θ, the designer's favorite outcomes are $f(\theta)$. While every player knows the state of the world, the designer does not. Think of, for example, a town hall (designer) trying to decide how much taxes to levy (outcomes) on a community of neighbors (players), where the optimal taxation depends on the productivities of different neighbors, a state of the world that every neighbor knows but the town hall does not.

Due to the designer's ignorance of θ, it does not know which outcome to pick and must proceed more indirectly. The goal of the designer is to come up with an incentive scheme, called a **mechanism**, that induces self-interested players to choose one of the designer's favorite outcomes. The mechanism enlists the help of the players, who know the state of the world, in selecting an outcome optimal from the point of view of the designer.

More precisely,

Definition 42 (Mechanism)**.** Given a mechanism design problem, a **mechanism** (S, g) consists of:

1. A set of (pure) strategies S_i for each $i \in N$.
2. A map $g : S \to A$.

A mechanism is a **game form**, i.e., a way to model the rules of a game, or an institution, independently of the players' utility functions over the game's outcomes. The designer announces a set of pure strategies S_i for each player and a mapping between the profile of pure strategies and the outcome. The designer promises to implement the outcome $g(s)$ when players choose the strategy profile s.

In state θ, the mechanism (S, g) **gives rise to a normal form game**, $G(\theta)$, where the set of actions of player i is S_i and the payoff i gets from strategy profile s is $u_i(g(s), \theta)$. The mechanism solves the designer's information problem if playing the game $G(\theta)$ yields the same outcomes as $f(\theta)$. To predict what agents will do when they play the game $G(\theta)$, the designer must **pick a solution concept**. We will use Nash equilibrium.

Let $\text{NE}_g(\theta)$ denote the set of **Nash equilibrium outcomes** in each state of the world,

$$\text{NE}_g(\theta) \equiv \{a \in A : \exists \text{ NE of } G(\theta), s, \text{ s.t. } a = g(s)\}.$$

Definition 43 (Nash implementation)**.** The mechanism (S, g) **Nash implements** social choice rule f if $\text{NE}_g(\theta) = f(\theta)$ for every $\theta \in \Theta$.

If the designer wants to use a solution concept other than Nash equilibrium, then it would simply replace "NE" in the above definition.

Loosely speaking, mechanism design is "**reverse game theory**". Whereas a game theorist takes the game as given and analyzes its equilibria, a mechanism designer takes the social choice rule as given and acts as a "game maker", aiming to engineer a game with suitable equilibria.

2.2 *Maskin monotonicity and no veto power.* It is natural to ask which mechanism design problems admit Nash implementations. As we saw in lecture the following pair of conditions are important.

Definition 44 (Maskin monotonicity)**.** A social choice rule f satisfies **Maskin monotonicity (MM)**[18] provided that for all $a \in A$ and $\theta, \theta' \in \Theta$, if

1. $a \in f(\theta)$,
2. for all $i \in N$ and $b \in A$, $u_i(a, \theta) \geq u_i(b, \theta) \Rightarrow u_i(a, \theta') \geq u_i(b, \theta')$,

then $a \in f(\theta')$.

Equivalently, we can write the second condition as for all player i, $\{b : u_i(a, \theta) \geq u_i(b, \theta)\} \subseteq \{b : u_i(a, \theta') \geq u_i(b, \theta')\}$. In words, if a is chosen in some state θ, then it should also be chosen when the set of outcomes weakly worse than a expands for everyone.

Definition 45 (No veto power)**.** A social choice rule f satisfies **no veto power (NVP)** provided that for all $a \in A$ and $\theta \in \Theta$, if there exists $i \in N$ such that

$$u_j(a, \theta) \geq u_j(b, \theta) \text{ for all } j \neq i \text{ and all } b \in A$$

then $a \in f(\theta)$.

Theorem 46 (Maskin, 1999)**.**

1. *If f is Nash implementable, then it satisfies MM.*
2. *If $n \geq 3$ and f satisfies MM and NVP, then f is Nash implementable.*

Proof. See lecture. □

Example 47 (NVP but not MM)**.** Suppose $n \geq 3$ and individuals have strict preferences over outcomes A in any state of the world. Consider the social choice rule "top-ranked rule", f^{Top}, that chooses the outcome(s) top-ranked by the largest number of individuals. That is, $a \in f^{\text{Top}}(\theta)$ if and only if for all $b \in A$,

$$\#\{i : u_i(a, \theta) > u_i(c, \theta) \text{ for all } c \neq a\} \geq \#\{i : u_i(b, \theta) > u_i(c, \theta) \text{ for all } c \neq b\}.$$

To see why f^{Top} satisfies NVP, suppose for $a \in A$ and $\theta \in \Theta$, there exists $i^* \in N$ such that $u_j(a, \theta) \geq u_j(b, \theta)$ for all $j \neq i^*$ and all $b \in A$. Since we assume preferences are strict, it follows that

$$\#\{i : u_i(a, \theta) > u_i(c, \theta) \text{ for all } c \neq a\} \geq n - 1,$$

[18] What Professor Maskin called "monotonicity" in lecture is usually referred to as "Maskin monotonicity" in the literature, cf. Footnote 8.

while for any $b \neq a$,

$$\#\{i : u_i(b,\theta) > u_i(c,\theta) \text{ for all } c \neq b\} \leq 1.$$

Since $n \geq 3$, $a \in f^{\text{Top}}(\theta)$.

To see why f^{Top} does not satisfy MM, consider the following preferences: $n = 3$, $A = \{a,b,c\}$. In state θ, $u_1(a,\theta) > u_1(b,\theta) > u_1(c,\theta)$, $u_2(b,\theta) > u_2(c,\theta) > u_2(a,\theta)$, and $u_3(c,\theta) > u_3(b,\theta) > u_3(a,\theta)$; in state θ', the preferences are unchanged except that $u_3(b,\theta') > u_3(c,\theta') > u_3(a,\theta')$. Then outcome a did not drop in ranking relative to any other outcome for any individual from θ to θ', yet $f^{\text{Top}}(\theta) = \{a,b,c\}$ while $f^{\text{Top}}(\theta') = \{b\}$. This shows f^{Top} does not satisfy MM. Hence by Theorem 46 it is not Nash implementable. ♦

Example 48 (MM but not NVP, yet implementable)**.** Suppose $n \geq 3$ and individuals have strict preferences over outcomes A in any state of the world. Consider the social choice rule "dictator's rule" f^{D}, that chooses the top-ranked outcome of player 1, the dictator.

To see why f^{D} satisfies MM, note that $a \in f(\theta)$ implies that $u_1(a,\theta) > u_1(b,\theta)$ for all $b \neq a$. In any state of the world θ' where a does not fall in ranking relative to any other outcome for any individual, it remains true that $u_1(a,\theta') > u_1(b,\theta')$ for any $b \neq a$. As such, $a \in f^{\text{D}}(\theta')$ also.

To see why f^{D} does not satisfy NVP, consider the following preferences: $n = 3$, $A = \{a,b\}$. In state θ, $u_1(a,\theta) > u_1(b,\theta)$, $u_2(b,\theta) > u_2(a,\theta)$, and $u_3(b,\theta) > u_3(a,\theta)$. We have b being top-ranked for all individuals except player 1, yet $f^{\text{D}}(\theta) = \{a\}$.

Theorem 46 does not say whether f^{D} is Nash implementable or not. However, it is easy to see that f^{D} can be implemented by the following mechanism: ask each player their favorite outcome, but only implements the answer of player 1 while ignoring everyone else. This example shows MM plus NVP are sufficient for Nash implementability when $n \geq 3$, but NVP is **not necessary.** ♦

Example 49 (MM but not NVP, and not implementable[19])**.** Suppose $n = 3$ and each state represents a set of strict orderings $(\succ_1, \succ_2, \succ_3)$ of all individuals over outcomes $A = \{a,b,c\}$. Consider the social choice rule, f:

- For $x \in \{a,b\}$, $x \in f(\theta)$ if and only if x is Pareto-optimal and top-ranked for 1.
- $c \in f(\theta)$ if and only if c is Pareto-optimal and not bottom-ranked for 1.

It is easy to verify that f satisfies MM. To see why f does not satisfy NVP, suppose that in state θ player 1 bottom-ranks c, then $c \notin f(\theta)$ even if players 2 and 3 top-rank c.

To see why f is not Nash implementable, consider the following three profiles $\theta, \theta', \theta''$:

- $\theta \;\; : b \succ_1 c \succ_1 a,\ c \succ_2 a \succ_2 b,\ c \succ_3 a \succ_3 b \quad \Rightarrow \quad f(\theta) = \{b,c\}.$
- $\theta' \;\; : a \succ_1 b \succ_1 c,\ c \succ_2 b \succ_2 a,\ c \succ_3 a \succ_3 b \quad \Rightarrow \quad f(\theta') = \{a\}.$
- $\theta'' : b \succ_1 a \succ_1 c,\ a \succ_2 b \succ_2 c,\ a \succ_3 b \succ_3 c \quad \Rightarrow \quad f(\theta'') = \{b\}.$

If f were implementable, there would exist a mechanism (S,g) and a Nash equilibrium s^* of $G(\theta)$ such that $g(s^*) = c$. It follows that for all $s_1 \in S_1$, $g(s_1, s_2^*, s_3^*) \neq b$, otherwise s_1 would be a profitable deviation for player 1 in state θ.

If there existed $s_1' \in S_1$ such that $g(s_1', s_2^*, s_3^*) = a$, then (s_1', s_2^*, s_3^*) would be a Nash equilibrium of $G(\theta'')$ (no one has a profitable deviation), a contradiction since $a \notin f(\theta'')$. We concluded that for all $s_1 \in S_1$, $g(s_1, s_2^*, s_3^*) = c$. But this indicates that s^* is a Nash equilibrium of $G(\theta')$, a contradiction since $c \notin f(\theta')$.

This example shows MM per se is **not sufficient** for Nash implementability. ♦

Example 50 (The electoral college rule[20])**.** Consider a society made up of three states, $\{A,B,C\}$. The voters in each state will vote over the set of candidates $\{H,T,J\}$. State A has 10 voters and 6 electors, state B has 7 voters and 5 electors, and state C has 3 voters and 2 electors. Once a candidate wins the state, the electors will vote according to the winning candidate in their state. Overall, the candidate wins who wins the most electoral votes.

Assume that in state of the world θ all 10 voters of state A have preferences $H \succ T \succ J$, all 7 voters of state B have preferences $T \succ H \succ J$, while in state C one voter has the preferences $H \succ J \succ T$, while the two remaining voters have $J \succ T \succ H$. Then H carries state A and has 6 electors, T carries state B and has 5 electors, while J carries state C and has 2 electors. Overall, candidate H wins with 6 electoral votes.

[19]This example is adapted from Maskin (1999, Example 2).

[20]This example is developed by Jetlir Duraj and Kevin He. They conjecture that the electoral college in the U.S. electoral system does not satisfy MM. This is an attempt at showing what could go wrong in a simple example.

Consider now state of the world θ', which is the same as θ except, that in state C the two last individuals have preferences $T \succ J \succ H$ (instead of $J \succ T \succ H$). Then H hasn't fallen relatively to the other candidates for any voter, but now candidate T wins the electoral college, by carrying states B and C with 7 electors in all.

This shows that the electoral college rule does not satisfy MM. Note, that by simple popular vote, H would win in both states of the world θ and θ', since she has 11 voters assured, while the most that T could hope to get is 9 votes (in θ').

Nevertheless, popular vote is also susceptible to failure of MM. You can try to think of an example. ♦

Example 51 (December 2016 Final Exam)**.** Suppose $n = 3$ and each state represents a set of strict orderings $(\succ_1, \succ_2, \succ_3)$ of all individuals over alternatives $A = \{a, b, c\}$. Let social choice rule f be "rank-order voting" ("Borda count"). That is, an alternative gets 3 points every time it is ranked first by some individual, 2 every time it is ranked second, and 1 point every time it is ranked third. Points are summed across individuals, and $f(\theta)$ consists of the alternative(s) with the highest overall point total. Prove that f is not implementable in Nash equilibrium.

Solution:

Let $R_i^\theta(a) \in \{1, 2, 3\}$ be the ranking alternative a gets from agent i in state θ. Then the rule we are considering is

$$f(\theta) = \left\{ a \in A : \sum_{i=1}^{3} R_i^\theta(a) \geq \sum_{i=1}^{3} R_i^\theta(b), \forall b \in A \right\}.$$

If f were Nash implementable, it would satisfy MM. Consider the state $\theta : a \succ_1 c \succ_1 b,\ b \succ_2 a \succ_2 c,\ c \succ_3 b \succ_3 a$. Then each of the alternatives in A gets 6 points, so $f(\theta) = \{a, b, c\}$. Pick $a \in f(\theta)$ and consider the state θ', where the preferences of players 2 and 3 are the same and player1's preference is, instead, $a \succ_1 b \succ_1 c$. Then $f(\theta') = \{b\}$ because b gets 7 points. This gives a contradiction to MM, as a hasn't fallen relatively to the other outcomes for any player. Therefore, f is not implementable. ♦

Economics 2010a ▹ Section 4: Bayesian Games — 11/14/2021

(1) Bayesian games; **(2)** Auction model; **(3)** Solving for auction BNEs; **(4)** Revenue equivalence theorem;

(5) Optional: The universal type space

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1 Bayesian Games

***1.1** The model of a Bayesian game.* In our brief encounter with mechanism design, we considered a setting where the designer is uncertain as to the state of the world $\theta \in \Theta$, but every player knows θ perfectly. Many economic situations involve uncertainty about payoff-relevant state of the world amongst even the **players themselves**. To take some examples:

- Auction participants are uncertain about other bidders' willingness to pay.
- Investors are uncertain about the profitability of a potential joint venture.
- Traders are uncertain about the value of a financial asset at a future date.

How should a group of **Bayesian** players confront such uncertainty?

Definition 52 (Bayesian game)**.** A **Bayesian game** $B = \langle N, (\Theta_i)_{i\in N}, (A_i)_{i\in N}, (u_i)_{i\in N}, (p_i)_{i\in N}\rangle$ consists of:

1. A (finite) collection of **players** $N = \{1, 2, ..., n\}$.
2. A set of **actions** A_i for each $i \in N$.
3. A set of **states of the world** $\Theta = \prod_{i=1}^{n} \Theta_i$.
4. A **prior** $p_i \in \Delta(\Theta)$ for each $i \in N$.
5. A **utility function** $u_i : A \times \Theta \to \mathbb{R}$ for each $i \in N$.

Definition 53. A **pure strategy** of player i in a Bayesian game is a mapping $s_i : \Theta_i \to A_i$. A **mixed strategy** of player i in a Bayesian game is a mapping $\sigma_i : \Theta_i \to \Delta(A_i)$.

While there exist some more general approaches (see the optional material on the universal type space, for example), most models of incomplete-information games you will encounter will impose the **common prior assumption**: there exists a **common prior** for each $i \in N$, which we call $\mu \in \Delta(\Theta)$.

For ease of exposition, for now we will focus on the case where Θ is finite.[21]

A Bayesian game proceeds as follows. A state of the world θ is realized. Player i learns the i-th dimension, θ_i, then takes a pure action from her action set A_i or a mixed action from $\Delta(A_i)$. The utility of player i depends on the profile of actions as well as the state of the world θ, so in particular it might depends on the dimensions of θ that i **does not observe**. The subset of Bayesian games where u_i **does not** depend on θ_{-i} are called **private value** games.

Player i's strategy is a function of θ_i, not of θ, for i can only condition her action on her partial knowledge of the state of the world. For reasons we make clear later, Θ_i is often called the **type space** of i and one often describes a strategy of i as "type θ_i' does X, while type θ_i'' does Y".

A strategy profile in a Bayesian game might remind you of a **correlated equilibrium**. Indeed, in both setups each player observes some realization (her signal in CE, her type in Bayesian game), then performs an action dependent on her observation. However, unlike (Ω, p) in the definition of a correlated equilibrium, the (Θ, p) in a Bayesian game is **part of the game**, not part of the solution concept. Furthermore, while the signal profile $\omega \in \Omega$ in a CE is only a coordination device that does not by itself affect players' payoffs (as in an unenforced traffic light), the state of the world in a Bayesian game is **payoff-relevant**.

[21] The Bayesian game model can also accommodate games with infinitely many states of the world, such as auctions with a continuum of possible valuations for each bidder.

Example 54 (August 2013 General Exam)**.** Two players play a game. With probability 0.5, the payoffs are given by the left payoff matrix. With probability 0.5, they are given by the right payoff matrix. Player 1 knows whether the actual game is given by the left or right matrix, while Player 2 does not. Model this situation as a Bayesian game.

	L	C	R
T	$-2,-2$	$-1,1$	$0,0$
M	$1,-1$	$3,5$	$3,4$
B	$0,0$	$4,2$	$2,4$

	L	C	R
T	$0,0$	$0,0$	$0,0$
M	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,0$	$4,4$

Solution:

Let $\Theta_1 = \{l, r\}$, $\Theta_2 = \{0\}$, $\mu \in \Delta(\Theta)$ with $\mu(l, 0) = \mu(r, 0) = 0.5$. In state $(l, 0)$, the payoffs are given by the left matrix. In state $(r, 0)$, they are given by the right matrix. There are thus two types of player 1: the type who knows that the payoffs are given by the left matrix, and the type who knows that the payoffs are given by the right one. There is only one type of player 2. The utility of each player depends on (a_1, a_2, θ). For example $u_2(B, C, (l, 0)) = 2$ while $u_2(B, C, (r, 0)) = 0$. In particular, the payoff to player 2 depends on θ_1, so this is **not a private value game**.

A pure strategy of player 1 in this Bayesian game is a function $s_1 : \Theta_1 \to A_1$, in other words the strategy must specify what the l-type and r-type of player 1 will do. A pure strategy of player 2 is a function $s_2 : \Theta_2 \to A_2$, but since Θ_2 is just a singleton, player 2 has just one action in each of her pure Bayesian game strategies. ♦

***1.2** Bayesian Nash equilibrium*. When a profile of pure opponent strategies s_{-i} is played, after observing θ_i, player i evaluates her expected utility from any pure action $a_i \in A_i$ by:

$$\mathbb{E}_i[u_i(a_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i] = \sum_{\theta_{-i}\in\Theta_{-i}} p_i(\theta_{-i}|\theta_i)u_i(a_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})).$$

Similarly, we can extend the domain into mixed strategies. When a profile of mixed opponent strategies σ_{-i} is played, after observing θ_i, player i evaluates her expected utility from any mixed action $\alpha_i \in \Delta(A_i)$ by:

$$\sum_{\theta_{-i}\in\Theta_{-i}} \sum_{a\in A} p_i(\theta_{-i}|\theta_i)\alpha_i(a_i)\left(\prod_{j\neq i}[\sigma_j(\theta_j)](a_j)\right)u_i(a_i, a_{-i}, (\theta_i, \theta_{-i})).$$

We will write $\mathbb{E}_i[u_i(\alpha_i, \sigma_{-i}(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i]$ for this utility. Here's the most common equilibrium concept for Bayesian games.

Definition 55 (Bayesian Nash equilibrium)**.** A **pure strategy Bayesian Nash equilibrium** is a pure strategy profile s^* such that for each player $i \in N$ and each type $\theta_i \in \Theta_i$,

$$\mathbb{E}_i[u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i] \geq \mathbb{E}_i[u_i(a_i', s_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i] \text{ for all } a_i' \in A_i.$$

Similarly, a **mixed strategy Bayesian Nash equilibrium (BNE)** is a mixed strategy profile σ^* such that for each player $i \in N$ and each type $\theta_i \in \Theta_i$,

$$\mathbb{E}_i[u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i] \geq \mathbb{E}_i[u_i(a_i', \sigma_{-i}^*(\theta_{-i}), (\theta_i, \theta_{-i}))|\theta_i] \text{ for all } a_i' \in A_i.$$

Note that in the definition of a mixed BNE, it is without loss of generality to require no profitable unilateral deviation to any **pure** action, a_i', rather than any **mixed** action.

A BNE might be understood as a "**correlated equilibrium with payoff-relevant signals**". Let's focus on a pure strategy BNE, s^*. After observing her type θ_i, player i derives from her prior a conditional belief $p_i(\cdot|\theta_i) \in \Delta(\Theta_{-i})$ about the types of other players. She knows $s_{-i}^*(\cdot)$, so she knows how these opponent types translate into opponent actions. Unlike in a CE, however, she knows that her payoff also depends on the complete state of the world, $\theta = (\theta_i, \theta_{-i})$. Analogous to CE, a BNE is a strategy profile such that, after player i observes her type θ_i and calculates her expected payoffs to different actions, she finds it optimal to play the prescribed action $s_i^*(\theta_i)$ across all of her choices in A_i.

Example 56 (August 2013 General Exam)**.** Find all the BNEs in Example 54.

Solution:

Since player 2 has only one type, it is easiest to break things down by player 2's action in equilibrium.

Step 1: Consider BNE where player 2 plays a pure strategy.

1. $s_2^* = L$. In any such BNE, we must have $s_1^*(l) = M$ since the type-l player 1 knows for sure that player 2 is playing L and that payoffs are given by the left matrix, leading to a unique best response of M. Yet this means player 2 has a profitable deviation. Playing C yields an expected payoff of $\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 0 = 2.5$ (regardless of what $s_1^*(r)$ is), which is better than playing L and getting an expected payoff of $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 0 = -0.5$. Therefore, there is no BNE with $s_2^* = L$.

2. $s_2^* = C$. In any such BNE, we must have $s_1^*(l) = s_1^*(r) = B$ for similar reasoning as above. But that means player gets an expected payoff of $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$ by playing C, yet he can get $\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$ by playing R. Therefore, there is no BNE with $s_2^* = C$.

3. $s_2^* = R$. In any such BNE, we must have $s_1^*(l) = M$, $s_1^*(r) = B$ for similar reasoning as above. As such, player 2 gets an expected payoff of $\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$ from playing R. By comparison, he would get an expected $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 0 = -0.5$ from playing L and an expected $\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 0 = 2.5$ from playing C.[22] Therefore, we see that $s_1^*(l) = M$, $s_1^*(r) = B$, $s_2^* = R$ is a BNE of the game.

Step 2: Consider BNE where player 2 mixes.

Note that in the left matrix, T is strictly dominated by M, so $[\sigma_1^*(l)](T) = 0$. In the right matrix, T and M are weakly dominated by B. Since we are ruling out the case of $\sigma_2^*(L) = 1$, it follows that $[\sigma_1^*(r)](B) = 1$.

$[\sigma_1^*(l)](T) = 0$ $[\sigma_1^*(r)](B) = 1$ together imply that for player 2, R yields strictly higher payoff than L in either matrix, so player 2 has to be mixing between C and R.

Suppose that $\sigma_1^*(l) = pM \oplus (1-p)B$. The indifference principle for player 2 implies that

$$0.5[5p + 2(1-p)] + 0.5 \cdot 0 = 0.5 \cdot 4 + 0.5 \cdot 4 \quad \Rightarrow \quad p = 2 > 1,$$

contradiction! So there is no such BNE.

To sum up, $s_1^*(l) = M$, $s_1^*(r) = B$, $s_2^* = R$ is the unique BNE of the game. ♦

Example 57 (From old problem sets of Jerry Green)**.** Two players are working together to complete a project. When players 1 and 2 choose effort levels $e_1, e_2 \in [0,1]$, the probability that the project is successfully completed is $\frac{1}{2}(1+e_1)e_2$. Assume that players receive payoff 1 if the project is successful, and that they incur a quadratic disutility of effort which makes their expected payoff in the game equal to

$$u_i(e_1, e_2; c_1, c_2) = \frac{1}{2}(1+e_1)e_2 - c_i e_i^2.$$

Assume that $c_2 = 1$ is common knowledge, but that c_1 is known only to player 1, with player 2's prior being that nature chooses c_1 according to the probability function,

$$f(x) = \frac{2}{5}x, \text{ if } x \in [2,3] \text{ and 0 otherwise.}$$

Find all (pure and mixed) BNE of this game.

Solution:

Player 1's type is given by his cost parameter $c_1 \in [2,3]$. Player 2 has a single type.

To find pure strategy BNE we note that in a BNE player 2's effort level will be fixed (a function of her single type), so player 1 can respond to it. Player 1's maximization problem is,

$$\max_{e_1} u_1(e_1, e_2; c_1, c_2) = \frac{1}{2}(1+e_1)e_2 - c_1 e_1^2.$$

This is strictly concave in e_1 and the maximum is attained at

$$BR_1(e_2; c_1) = \frac{e_2}{4c_1}.$$

In contrast, player 2 must play a best response *given expectations*, since in equilibrium she has uncertainty about player 1's effort level. Her maximization problem is

$$\max_{e_2} \mathbb{E}_{e_1}[u_2] = \mathbb{E}_{e_1}\left[\frac{1}{2}(1+e_1)e_2 - e_2^2\right] = \frac{1}{2}(1+\mathbb{E}[e_1])e_2 - e_2^2.$$

[22] It is not feasible for player 2 to "play C in the left matrix, play R in the right matrix" since he can only condition his action on his type. Player 2 knows only the prior probabilities of the two matrices, but not which one is actually being played.

This is again strictly concave in e_2 and the maximum is attained at

$$BR_2(e_1) = \frac{1 + \mathbb{E}[e_1]}{4}.$$

In a BNE the strategies $\left(e_1^*(c_1), e_2^*\right)$ are best responses to each other. That is

$$e_1^*(c_1) = \frac{e_2^*}{4c_1}, \quad e_2^* = \frac{1 + \mathbb{E}\left[e_1^*(c_1)\right]}{4}.$$

To solve we calculate first

$$\mathbb{E}[e_1^*(c_1)] = \int_2^3 \left(\frac{e_2^*}{4c_1}\right)\frac{2}{5}c_1\, dc_1 = \frac{e_2^*}{10}\int_2^3 dc_1 = \frac{e_2^*}{10}.$$

Plugging this into the best response function for player 2 yields

$$e_2^* = \frac{1 + \frac{e_2^*}{10}}{4}.$$

Solving we find $e_2^* = \frac{10}{39}$. Plugging this into the equation of player 1 we get

$$e_1^*(c_1) = \frac{10}{4 \cdot 39 c_1} = \frac{5}{78 c_1}.$$

Since each player's maximization problem is strictly concave, the best response correspondences are single-valued (for fixed strategy of the other player). From this it follows there are no mixed strategy BNEs.

In all, the unique BNE is

$$(e_1^*(c_1), e_2^*) = \left(\frac{5}{78c_1}, \frac{10}{39}\right).$$

♦

Example 58 (From MWG)**.** The Alphabeta research and development consortium has two (non-competing) members, firms 1 and 2. The rules of the consortium are that any independent invention by one of the firms is shared fully with the other. Suppose that there is a new invention, the 'Zigger', that either of the two firms could potentially develop. To develop this new product costs $c \in (0, 1)$. The benefit of the Zigger to each firm is known only to that firm. Formally, each firm i has a type θ_i that is independently drawn from a uniform distribution over [0,1], and its benefit in case of type θ_i is θ_i^2. The timing is as follows: The two firms privately observe their type. Then they each both simultaneously decide to develop or not. Find the pure BNE of this game.

Solution:

We write $s_i(\theta_i) = 1$ if firm i develops and $s_i(\theta_i) = 0$ otherwise. If firm i develops when her type is θ_i then her payoff is $\theta_i^2 - c$, regardless of the action of the other firm. If firm i decides to not develop when her type is θ_i her expected payoff is $\theta_i^2 \Pr_{\theta_j}\left(s_j\left(\theta_j\right) = 1\right)$. Hence, one calculates easily that developing is a best response for type θ_i of i if and only if

$$\theta_i \geq \left[\frac{c}{1 - \Pr_{\theta_j}\left(s_j\left(\theta_j\right) = 1\right)}\right]^{\frac{1}{2}}.$$

This inequality shows that the strategies in any potential BNE take the form of a cut-off rule: develop if and only if own type is higher than a threshold. Let $\overline{\theta}_i$, $i = 1, 2$ be the cut-offs in a BNE. Given the cutoff strategies, we have $\Pr_{\theta_j}\left(s_j\left(\theta_j\right) = 1\right) = 1 - \overline{\theta}_j$ so that the thresholds satisfy the equations

$$\overline{\theta}_1\left(\overline{\theta}_2\right)^2 = c, \quad \overline{\theta}_2\left(\overline{\theta}_1\right)^2 = c.$$

This implies that $\overline{\theta}_1 = \overline{\theta}_2 = c^{\frac{1}{3}}$. This gives a unique potential BNE. It is then straightforward to check that the threshold strategies with threshold equal to $c^{\frac{1}{3}}$ are indeed a BNE. ♦

Example 59 (Purification of mixed strategies)**.**

1. Find the unique Nash equilibrium of the following normal form game.

	L	R
T	$0, 0$	$0, -1$
B	$1, 0$	$-1, 3$

2. Consider now the following perturbed version where $\varepsilon > 0$ is a small number and a, b are independent and uniformly distributed in $[0, 1]$.

	L	R
T	$\varepsilon a, \varepsilon b$	$\varepsilon a, -1$
B	$1, \varepsilon b$	$-1, 3$

   Assume player 1 sees the draw of a but not of b and player 2 sees the draw of b but not of a. Find the essentially unique Bayesian Nash equilibrium for fixed ε.

3. What happens as ε goes to zero? Interpret.

Solution:

1. It easy to see that there is no Nash equilibrium in pure strategies. It is also easy to see that there cannot be a Nash equilibrium where only one of the players strictly mixes. By using the usual indifference conditions we arrive at the unique Nash equilibrium where both player strictly mix $\left(\frac{3}{4}T \oplus \frac{1}{4}B; \frac{1}{2}L \oplus \frac{1}{2}R\right)$.

2. Note that for a fixed strategy of player 2, as a becomes larger and larger, T becomes more and more attractive in comparison to B. There is thus a threshold value for a, call it p, so that player 1 chooses T whenever $a > p$ and B whenever $a < p$. The same logic applies to player 2 and thus there is a threshold q for him so that L is chosen if $b > q$ and R if $b < q$.

   From the perspective of player 1, at the threshold p the two pure strategies should be indifferent. Given the threshold strategy of player 2 this implies that for $a = p$

$$\varepsilon p = 1 \cdot (1 - q) + (-1) \cdot q.$$

   A similar indifference condition holds for player 2:

$$\varepsilon q = -1 \cdot (1 - p) + 3 \cdot p.$$

   We can solve this system in the two unknowns (p, q) to get

$$(p, q) = \left(\frac{2 + \varepsilon}{8 + \varepsilon^2}, \frac{4 - \varepsilon}{8 + \varepsilon^2}\right).$$

   Thus, any BNE satisfies

$$s_1(a) = \begin{cases} T, & \text{if } a > \frac{2+\varepsilon}{8+\varepsilon^2}, \\ B, & \text{if } a < \frac{2+\varepsilon^2}{8+\varepsilon^2}, \end{cases} \qquad s_2(b) = \begin{cases} L, & \text{if } b > \frac{4-\varepsilon}{8+\varepsilon^2}, \\ R, & \text{if } b < \frac{4-\varepsilon}{8+\varepsilon^2}. \end{cases}$$

   To fully specify a BNE we need to specify strategies for the cases where $a = p$ and $b = q$. We can take any possible specification of strategies for these cases, because they happen with probability zero from the perspective of the other player[23], and the relevant optimality calculations involve comparing expectations/averages which don't depend on changes made in zero-probability events. Thus, up to strategies picked for the cases $a = p$ and $b = q$, the BNE strategies are unique.

[23] a is distributed according to a continuous distribution from the perspective of player 2 and similarly for b and player 1.

3. Note that as ε goes to zero, (p, q) converges to $\left(\frac{1}{4}, \frac{1}{2}\right)$. These thresholds give precisely the mixed NE calculated in part 1. These kinds of perturbation arguments to support mixed Nash equilibria are well-known since seminal work from Harsanyi who was trying to 'micro-found' why players in a game as in part 1, which has a unique mixed NE, would pick the 'right' probabilities to randomize with, given their indifference between the pure strategies in equilibrium. One possible story to tell is thus, that the players perceive the game as 'perturbed' as above and are actually playing a pure BNE in a Bayesian game whose play converges to the mixed Nash of the unperturbed game, as the perturbation goes to zero.

♦

2 Auction Model

***2.1** Definition of an auction.* In section, we will make a number of simplifying assumptions instead of studying the most general auction model. We will assume that: (i) auctions are in **private value**, so the types of $-i$ do not matter for i's payoff; (ii) the type distribution is **identical** and **independent** across players; (iii) there is one seller who sells **one indivisible item**; (iv) players are **risk neutral**, so getting the item with probability H and having to pay P in expectation gives a player i with type θ_i a utility of $\theta_i H - P$.

Definition 60 (Auction)**.** An **auction** $\left\langle N, F, \left[0, \bar{\theta}\right], (H_i)_{i\in N}, (P_i)_{i\in N}\right\rangle$ consists of:

1. A (finite) set of **bidders** $N = \{1, ..., n\}$.
2. A **type distribution** F over $\left[0, \bar{\theta}\right]$, which admits a continuous density f with $f(\theta_i) > 0$ for all $\theta_i \in \left[0, \bar{\theta}\right]$.
3. An **allocation rule** $H_i : \mathbb{R}^n_+ \to [0, 1]$ that specifies the probability that bidder i gets the item for every profile of n bids for each $i \in N$.
4. A **payment rule** $P_i : \mathbb{R}^n_+ \to \mathbb{R}$ that specifies the expected payment of bidder i for every profile of n bids for each $i \in N$.

At the start of the auction, each player i learns her own valuation θ_i. The valuations of different players are drawn i.i.d. from F, which is supported on the interval $\left[0, \bar{\theta}\right]$. Each player simultaneously submits a nonnegtive real number as her bid. When the profile of bids $(b_1, ..., b_n)$ is submitted, player i gets the item with probability $H_i(b_1, ..., b_n)$ and pays $P_i(b_1, ..., b_n)$ in expectation.

***2.2** Some examples of (H, P) pairs.* In lecture, we showed that a number of well-known auction formats – namely, first-price and second-price auction – can be written in terms of some (allocation rule, payment rule) pairs. Now, we turn to a number of unusual auctions to further illustrate the definition.[24]

- **Raffle.** Each player chooses how many raffle tickets to buy. Each ticket costs \$1. A winner is selected by drawing a raffle ticket at random. This corresponds to $H_i(b_1, ..., b_n) = \frac{b_i}{\sum_k b_k}$, $P_i(b_1, ..., b_n) = b_i$. Unlike the usual auction formats like first-price and second-price auctions, the allocation rule H_i involves **randomization** for almost all profiles of "bids".
- **War of attrition.** A strategic territory is contested by two generals. Each general chooses how much resources to use in fighting for this territory. The general who commits more resources destroys all of her opponents' forces and wins the territory, but suffers as much losses as the losing general. This corresponds to

$$H_i(b_i, b_{-i}) = \begin{cases} 1, & \text{if } b_i > b_{-i}, \\ 0.5, & \text{if } b_i = b_{-i}, \\ 0, & \text{if } b_i < b_{-i}, \end{cases}$$

  and $P_i(b_i, b_{-i}) = \min\{b_1, b_2\}$, so it is as if two bidders each submits a bid and everyone **pays the losing bid**.
- **All-pay auction.** Each player submits a bid and the highest bidder gets the item. Every player, win or lose, must **pay her own bid**. Here, $H_i(b_1, ..., b_n)$ is the same as in first-price auction, but the payment rule is $P_i(b_1, ..., b_n) = b_i$.

[24]Some of these examples may seem to have nothing to do with auctions at a first glance, yet our definition of an auction in terms of (H, P) is general enough to apply to them. Here, as elsewhere in economics, theory allows us to unify our modeling and understanding of seemingly disparate phenomena under a single framework.

2.3 Auctions as private-value Bayesian games. Auctions form an **important class of examples in Bayesian games**. As defined above, an auction is a private value Bayesian game with a continuum of types for each player. Referring back to Definition 52, an auction $\left\langle N, F, \left[0, \bar{\theta}\right], (H_i)_{i\in N}, (P_i)_{i\in N}\right\rangle$ can be viewed as a common prior private value Bayesian game $B = \langle N, (\Theta_i)_{i\in N}, (A_i)_{i\in N}, (u_i)_{i\in N}, \mu\rangle$, where:

1. N is the set of players.
2. Player i's action set is $A_i = \mathbb{R}_+$, where actions $b_i \in \mathbb{R}_+$ are interpreted as bids.
3. States of the world is $\Theta = \left[0, \bar{\theta}\right]^n$, where the i-th dimension is the valuation of player i.
4. The common prior μ on Θ is the product distribution on $\left[0, \bar{\theta}\right]^n$ derived from F.
5. Utility function u_i specifies $u_i(b_1, ..., b_n, \theta_i) = \theta_i H_i(b_1, ..., b_n) - P_i(b_1, ..., b_n)$.

As such, many terminologies from general Bayesian games carry over to auctions. A pure strategy of bidder i is a function $s_i : \Theta_i \to \mathbb{R}_+$, mapping i's valuation to a nonnegative bid. A (pure strategy) BNE in an auction is a strategy profile s^* such that for each player i and valuation $\theta_i \in \Theta_i$,

$$s_i^*(\theta_i) \in \arg\max_{b_i\in\mathbb{R}_+} \mathbb{E}_{\theta_{-i}}\left[\theta_i H_i(b_i, s_{-i}^*(\theta_{-i})) - P_i(b_i, s_{-i}^*(\theta_{-i}))\right].$$

As usual, player i of type θ_i knows the **mapping** from opponent's types θ_{-i} to their actions $s_{-i}^*(\theta_{-i})$, i.e., how each opponent would bid as a function of their valuation, but she **does not know opponents' realized valuations**. She does know the **distribution** over opponents' valuations, so she can compute the expected payoff of playing different bids, with expectation[25] taken over opponents' types.

3 Solving for Auction BNEs

Given an auction, here are two approaches for identifying some of its BNEs. But be warned: an auction may have multiple BNEs and the following methods may not find all of them.

3.1 *Weakly dominant BNEs.* The following holds in general for private-value Bayesian games.

Definition 61 (Weakly dominant)**.** In a private value Bayesian game, a strategy $s_i : \Theta_i \to A_i$ is **weakly dominant** for player i if for all $a_{-i} \in A_{-i}$ and all $\theta_i \in \Theta_i$,

$$u_i(s_i(\theta_i), a_{-i}, \theta_i) \geq u_i(a_i', a_{-i}, \theta_i) \text{ for all } a_i' \in A_i.$$

Proposition 62. *In a private value Bayesian game, consider a strategy profile s^* where for each $i \in N$, s_i^* is weakly dominant for i. Then s^* is a BNE.*

Proof. For each $i \in N$ and $\theta_i \in \Theta_i$, definition of weakly dominant strategy says

$$u_i(s_i^*(\theta_i), a_{-i}, \theta_i) \geq u_i(a_i', a_{-i}, \theta_i) \text{ for all } a_i' \in A_i \text{ and } a_{-i} \in A_{-i}.$$

So in particular,

$$u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i) \geq u_i(a_i', s_{-i}^*(\theta_{-i}), \theta_i) \text{ for all } a_i' \in A_i \text{ and } \theta_{-i} \in \Theta_{-i}.$$

Taking expectation over $\theta_{-i} \in \Theta_{-i}$, we get

$$\mathbb{E}_i[u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i)|\theta_i] \geq \mathbb{E}_i[u_i(a_i', s_{-i}^*(\theta_{-i}), \theta_i)|\theta_i] \text{ for all } a_i' \in A_i.$$

This is just the definition of a BNE. □

As a result, if we can identify a weakly dominant strategy for each player in an auction, then a profile of such strategies forms a BNE.

Example 63 (Second-price auction with reserve price)**.** The seller sets **reserve price** $r \in \mathbb{R}_+$, then every bidder submits a bid simultaneously.

[25] This is analogous to Definition 55. However, in Definition 55 we spelled out a weighted sum over $\theta_{-i} \in \Theta_{-i}$ instead of writing an expectation. This was possible since we focused on the case of a finite Θ in that section.

- If every bid is less than r, then no bidder gets the item and no one pays anything.
- If the highest bid is r or higher, then the highest bidder gets the item and pays either the bid of the second highest bidder or r, whichever is larger. If several players tie for the highest bidder, then one of these high bidders is chosen uniformly at random, gets the item, and pays the second highest bid (which is equal to her own bid).

We argue that $s_i^*(\theta_i) = \theta_i$ is a weakly dominant strategy.

If $\theta_i < r$, then against any profile of opponent bids b_{-i}, $u_i(\theta_i, b_{-i}, \theta_i) = 0$ since i will never win the item from a bid less than the reserve price. Yet, any other bid can only get payoff no larger than 0, since any bid that wins must be larger than r, which is larger than i's valuation.

If $\theta_i \geq r$, then profiles of opponent bids b_{-i} may be classified into 3 cases.

Case 1: Highest rival bid is $y \geq \theta_i$. Then $u_i(\theta_i, b_{-i}, \theta_i) = 0$, while any other bid either loses the item or wins the item at a price of y, which can only lead to non-positive payoffs in the event of winning.

Case 2: Highest rival bid is $y \in [r, \theta_i)$. Then $u_i(b_i, b_{-i}, \theta_i) = \theta_i - y > 0$ for any $b_i > y$ (including $b_i = \theta_i$), since all bids higher than y lead to winning the item at a price of y. Bidding y leads to an expected payoff no larger than $\frac{1}{2}(\theta_i - y)$ from tie-breaking, which is worse than $\theta_i - y$. Bidding less than y loses the item and gets 0 utility.

Case 3: Highest rival bid is $y < r$. Then $u_i(b_i, b_{-i}, \theta_i) = \theta_i - r > 0$ for any $b_i \geq r$ (including $b_i = \theta_i$). Bidding less than the reserve price r loses the item and gets 0 utility.

Therefore, we have verified that playing $s_i^*(\theta_i) = \theta_i$ is optimal for type θ_i, regardless of opponents' bid profile. This means bidding own valuation is weakly dominant. By Proposition 62, every player bidding own valuation is therefore a BNE. ♦

3.2 The FOC approach. In the BNE of an auction, fixing bidder i with type θ_i, we have:

$$s_i^*(\theta_i) \in \arg\max_{b_i \in \mathbb{R}_+} \mathbb{E}_{\theta_{-i}} \left[\theta_i H_i(b_i, s_{-i}^*(\theta_{-i})) - P_i(b_i, s_{-i}^*(\theta_{-i}))\right]. \tag{2}$$

So in particular,

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} \mathbb{E}_{\theta_{-i}} \left[\theta_i H_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})) - P_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i}))\right] \tag{3}$$

because (3) **restricts** the optimization problem in (2) to the domain of $s_i^*(\Theta_i) \subseteq \mathbb{R}_+$. Essentially, condition (3) converts the problem of choosing an optimal bidding strategy to the problem of choosing a type to report. There could exist other best responses, but we require truth telling to be one of them. More generally, the **revelation principle** implies that for any BNE of any auction game, there exists an equivalent BNE of a **direct revelation mechanism** in which players announce types as strategies, and, in equilibrium, report their true types.

Consider now the objective function of this second optimization problem,

$$U_i(\hat{\theta}_i, \theta_i) \equiv \mathbb{E}_{\theta_{-i}} \left[\theta_i H_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})) - P_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i}))\right]. \tag{4}$$

If it is differentiable (which will hold provided H_i, P_i, and the distribution F are "nice enough") and valuation $\theta_i \in \left(0, \overline{\theta}\right)$ is interior, then the **first-order condition (FOC)** of optimization implies

$$\frac{\partial U_i}{\partial \hat{\theta}_i}(\theta_i, \theta_i) = 0.$$

In auctions without a weakly dominant strategy, sometimes this FOC can help identify a BNE by giving us a closed-form expression of $s_i^*(\theta_i)$ after manipulation.

Example 64 (First-price auction)**.** Consider a first-price auction with two bidders. The two bidders' types are distributed i.i.d. with $\theta_i \sim U[0, 1]$. Each bidder submits a nonnegative bid and whoever bids higher wins the item and pays her own bid. If there is a tie, then each bidder gets to buy the item at her bid with equal probability. It is known that this auction has a symmetric BNE (s_1^*, s_2^*) where (i) $s_i^*(\theta_i)$ is differentiable, strictly increasing in θ_i; (ii) the associated equation (4) is differentiable. Find a closed-form expression for $s_i^*(\theta_i)$.

Solution:

In the symmetric BNE (s_1^*, s_2^*), the expected probability of player 1 winning the item by playing the BNE strategy of type $\hat{\theta}_1$ is $\hat{\theta}_1$. This is because s_2^* is strictly increasing and symmetric to s_1^*, so that bidding $s_1^*(\hat{\theta}_1)$ wins is exactly when $\theta_2 < \hat{\theta}_1$, which happens with probability $\hat{\theta}_1$ since $\theta_2 \sim U[0, 1]$. At the same time, the expected payment for submitting the BNE bid of type $\hat{\theta}_i$ is $\hat{\theta}_i s_i^*(\hat{\theta}_i)$, because bidding $s_i^*(\hat{\theta}_i)$ wins with probability $\hat{\theta}_i$ and pays $s_i^*(\hat{\theta}_i)$ in the event of winning. The relevant optimization problem is therefore

$$\max_{\hat{\theta}_i \in [0,1]} \theta_i \hat{\theta}_i - \hat{\theta}_i s_i^*(\hat{\theta}_i).$$

FOC implies that

$$\theta_i - s_i^*(\theta_i) - \theta_i \frac{ds_i^*}{d\theta_i}(\theta_i) = 0.$$

This is a first-order differential equation in θ_i that holds for $\theta_i \in (0, 1)$, which yields

$$\frac{1}{2}\theta_i^2 + C = \theta_i s_i^*(\theta_i). \tag{5}$$

Evaluating at $\theta_i = 0$ recovers[26] the constant of integration $C = 0$. Therefore, $s_i^*(\theta_i) = \theta_i/2$ is the desired symmetric BNE. ♦

4 Revenue Equivalence Theorem

4.1 *The revenue equivalence theorem.* While you may be familiar with statements like "first-price auction and second-price auction are revenue equivalent" before taking this course, it is important to gain a more precise understanding of the **revenue equivalence theorem (RET)**. To see how a cursory reading of the RET might lead you astray, consider the **asymmetric** second-price auction BNE from lecture, where bidder 1 always bids $\bar{\theta}$ and everyone else always bids 0, regardless of their types. The seller's expected revenue is 0!

Strictly speaking, RET is not a statement comparing two auction formats, but a statement **comparing two equilibria of two auction formats**. "Revenue" is an equilibrium property and an auction game might admit multiple BNEs with different expected revenues.

So let a BNE s^* of some auction game be given.[27] Let us define two functions $G_i, R_i : \Theta_i \to \mathbb{R}$ for each player i, so that $G_i(\hat{\theta}_i)$ and $R_i(\hat{\theta}_i)$ give the **expected probability of winning** and **expected payment** when bidding as though valuation is $\hat{\theta}_i$:

$$G_i(\hat{\theta}_i) \equiv \mathbb{E}_{\theta_{-i}}\left[H_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i}))\right],$$
$$R_i(\hat{\theta}_i) \equiv \mathbb{E}_{\theta_{-i}}\left[P_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i}))\right].$$

The expectations are taken over opponents' types. Importantly, G_i and R_i are dependent on the choice of BNE s^*. If we consider a different BNE of the same auction, then we will have a different pair $(\tilde{G}_i, \tilde{R}_i)$.

To illustrate, consider the symmetric BNE we derived in the two-player auction in Example 64, where $s_i^*(\theta_i) = \theta_i/2$. It should be intuitively clear that $G_i(\hat{\theta}_i) = \hat{\theta}_i$ and $R_i(\hat{\theta}_i) = \hat{\theta}_i \cdot \frac{\hat{\theta}_i}{2} = \frac{\hat{\theta}_i^2}{2}$. We can also derive these expressions by definition:

$$G_1(\hat{\theta}_1) = \int_0^1 H_1\left(\frac{\hat{\theta}_1}{2}, \frac{\theta_2}{2}\right) d\theta_2 = \int_0^{\hat{\theta}_1} 1\, d\theta_2 + \int_{\hat{\theta}_1}^1 0\, d\theta_2 = \hat{\theta}_1,$$
$$R_1(\hat{\theta}_1) = \int_0^1 P_1\left(\frac{\hat{\theta}_1}{2}, \frac{\theta_2}{2}\right) d\theta_2 = \int_0^{\hat{\theta}_1} \frac{\hat{\theta}_1}{2}\, d\theta_2 + \int_{\hat{\theta}_1}^1 0\, d\theta_2 = \frac{\hat{\theta}_1^2}{2}.$$

As we have seen in lecture, the celebrated RET is just a corollary of the following result:

Proposition 65. *Fix a BNE s^* of the auction game. Under regularity conditions, $R_i(\theta_i) = \int_0^{\theta_i} xG_i'(x)\,dx + R_i(0)$ for all bidder i and type θ_i.*

Proof. See lecture. □

This result expresses the expected payment of an **arbitrary type** of bidder i in a BNE as a function of: (i) expected payment of the **lowest type** of bidder i in this BNE; (ii) the **expected probabilities of winning** for various types of player i in this BNE. It then follows that:

[26] Even though the FOC only applies for interior $\theta_i \in (0, 1)$, continuity of s_i^* implies that equation (5) holds even at the boundary points. This is sometimes called "value matching".

[27] We can in fact define G_i and R_i for any arbitrary profile of strategies s, without imposing that it is a BNE. However, Proposition 65 only holds when s is a BNE.

Theorem 66 (Revenue equivalence theorem)**.** *Under regularity conditions, for two BNEs of two auctions such that*

1. $G_i(\theta_i) = G_i^\circ(\theta_i)$ *for all* i *and* θ_i,
2. $R_i(0) = R_i^\circ(0)$ *for all* i,

then $R_i(\theta_i) = R_i^\circ(\theta_i)$ *for all* i *and* θ_i.

This follows directly from Proposition 65. Since in a BNE, the expected payment of an arbitrary type is entirely determined by the winning probabilities of different types and the expected payment of the lowest type, two BNEs where these two objects match must have the same expected payment for all types.

Here are two examples where RET is not applicable due to G_i and G_i° not matching up for two BNEs.

Example 67. In a second-price auction, the asymmetric BNE does not satisfy the conditions of RET when compared to the symmetric BNE of bidding own valuation. In the asymmetric equilibrium where where bidder 1 always bids $\overline{\theta}$ and everyone else always bids 0, $G_i(\theta_i) = 0$ for all $i \neq 1$ and all θ_i, since bidders other than 1 never win. Therefore, we cannot conclude from RET that these two BNEs yield the same expected revenue. (In fact, they do not.) ♦

Example 68. In Example 63, we showed that bidding own valuation is a BNE in a second-price auction with reserve price. When reserve price is $r > 0$, this BNE does not satisfy the conditions of RET when compared to the BNE of bidding own valuation in a second-price auction **without** reserve price. In the former BNE, $G_i(\theta_i) = 0$ for any $\theta_i \in (0, r)$, whereas in the latter BNE these types have a strictly positive probability of winning the item. Therefore, we cannot conclude from RET that these two BNEs in two auction formats yield the same expected revenue.

In fact, different reserve prices may lead to different expected revenues. Myerson (1981) tells you how to pick optimal reserve prices to maximize the expected revenue of an auction. ♦

4.2 Using RET to solve auctions. Sometimes, we can use RET to derive a closed-form expression of the BNE strategy profile s^*.

Example 69. As in Example 64, consider a first-price auction with two bidder whose valuations are i.i.d. with $\theta_i \sim U[0, 1]$. Assume this auction has a symmetric BNE where $s_i^*(\theta_i)$ strictly increasing in θ_i. Then this BNE is revenue equivalent to the BNE of second-price auction where each player bids own valuation. To see this, since both BNEs feature strategies strictly increasing in type, i of type θ_i wins precisely when player $-i$ has a type $\theta_{-i} < \theta_i$. That is to say, $G_i(\theta_i) = \theta_i = G_i^\circ(\theta_i)$. At the same time, the expected payment from type 0 is 0 in both BNEs – in particular, the type 0 bidder in first-price auction never wins since bids are strictly increasing in type, so never pays anything.

But in the bid-own-valuation BNE of the second-price auction, $R_i^\circ(\theta_i) = \theta_i \cdot (\theta_i/2)$, where θ_i is probability of being the highest bidder and $\theta_i/2$ is expected rival bid in the event of winning. By RET, $R_i(\theta_i) = \theta_i \cdot (\theta_i/2)$ also. In first-price auction, i pays own bid $s_i^*(\theta_i)$ whenever she wins, which happens with probability θ_i. Hence $s_i^*(\theta_i) = R_i(\theta_i)/\theta_i = \frac{\theta_i}{2}$. This is the same as what we found using FOC in Example 64. ♦

While in the above example we used RET to verify a result we already knew from FOC, RET can also be used in lieu of FOC to find BNEs. This can be particularly helpful when the differential equation from the FOC approach is harder to solve.

Example 70 (December 2012 Final Exam)**.** Suppose there are two risk-neutral potential buyers of an indivisible good. It is common knowledge that each buyer i's valuation is drawn independently from the same distribution on [0, 1] with distribution function $F(\theta) = \theta^3$, but the realizations of the θ_i's are private information. Calculate the expected payment $R_i(\theta_i)$ that a buyer with reservation price θ_i makes in the unique symmetric equilibrium of a **second**-price auction. Then, using the revenue equivalence theorem, find the equilibrium bid function in a **first**-price auction in the same setting.

Solution:

In the second-price auction, it is a weakly dominant BNE to bid own valuation (regardless of underlying distribution). In this BNE,

$$R_i(\theta_i) = \int_0^{\theta_i} \theta_j f(\theta_j)\, d\theta_j = \int_0^{\theta_i} \theta_j (3\theta_j^2)\, d\theta_j = \frac{3}{4}\theta_j^4 \Big|_{\theta_j=0}^{\theta_j=\theta_i} = \frac{3}{4}\theta_i^4.$$

This symmetric BNE of second-price auction is revenue equivalent to any BNE in first-price auction where bid increases strictly with own type. This is because in these BNEs, $G_i(\theta_i) = G_i^\circ(\theta_i) = \theta_i^3$ (since i of type θ_i wins exactly when $-i$ is of type lower than θ_i) and $R_i(0) = R_i^\circ(0) = 0$ (since type 0 never wins, so never pays). But in first-price auction, $R_i^\circ(\theta_i) = s_i^*(\theta_i) G_i^\circ(\theta_i)$, so then $s_i^*(\theta_i) = \left(\frac{3}{4}\theta_i^4\right)/\theta_i^3 = \frac{3}{4}\theta_i$. ♦

Example 71 (December 2016 Final Exam)**.** Consider the high-bid auction with one indivisible good for sale assuming the seller sets a reserve price of $\frac{1}{2}$. There are two risk-neutral buyers whose valuations are drawn independently from a distribution on $[0, 1]$ with c.d.f. $F(t) = t^2$. Find the buyers' bid function in the symmetric BNE of this auction game. Don't forget to justify your answer.

Solution:

In Example 63, we showed that bidding own valuation is a BNE in a second-price auction with reserve price. Thus, $s^\circ(\theta) = \theta$, $\theta \in [0, 1]$ is a symmetric BNE of the second-price auction with reservation price $\frac{1}{2}$ (henceforth SPAr). We conjecture that there is a symmetric BNE of the first-price auction with reservation price $\frac{1}{2}$ (henceforth FPAr) with $s(\theta) = 0$ when $\theta_i < \frac{1}{2}$ and $s(\theta)$ strictly increasing for $\theta \in \left[\frac{1}{2}, 1\right]$.

Note that in both BNE, type $\theta = 0$ pays 0 in expectation: $R^\circ(0) = R(0) = 0$. Furthermore, in both BNE we have for the expected probability of winning $G^\circ(\theta) = G(\theta) = 0, \theta \in \left[0, \frac{1}{2}\right)$ and $G^\circ(\theta) = G(\theta) = \theta^2, \theta \in \left[\frac{1}{2}, 1\right]$. In all, the conditions for applying RET are satisfied. We calculate for the case of the BNE of SPAr: $R^\circ(\theta) = 0$ for $\theta < \frac{1}{2}$ and for $\theta \geq \frac{1}{2}$

$$R^\circ(\theta) = \int_0^{\frac{1}{2}} \frac{1}{2} \cdot 2y\, dy + \int_{\frac{1}{2}}^{\theta} y \cdot 2y\, dy + \int_{\theta}^{1} 0 \cdot 2y\, dy = \frac{2}{3}\theta^3 + \frac{1}{24}.$$

Meanwhile, we have $R(\theta) = \theta^2 s(\theta)$ for FPAr. Equating $R^\circ(\theta) = R(\theta)$, we get

$$s(\theta) = \begin{cases} 0, & \text{if } \theta < \frac{1}{2}, \\ \frac{2}{3}\theta + \frac{1}{24\theta^2}, & \text{if } \theta \geq \frac{1}{2}. \end{cases}$$

We check by calculating, that indeed $s'(\theta) = \frac{2}{3} - \frac{1}{12\theta^3} > 0$, which is true for $\theta > \frac{1}{2}$ (precisely in the region where we need it).

Finally, we check that the candidate is indeed a BNE of FPAr. It is obvious that types $\theta \leq \frac{1}{2}$ don't have any profitable deviation: if they bid so as to have positive probability of winning the action they will have a negative expected payoff, which is worse than 0 they are getting in equilibrium. We focus now on types $\theta > \frac{1}{2}$. They wouldn't deviate to some $b > s(1)$, which is the highest possible rival bid. If they were to bid $b \leq \frac{1}{2}$, then payoff would be zero, while equilibrium payoff is

$$\theta^2 \cdot \theta - \theta^2 \cdot \left(\frac{2}{3}\theta + \frac{1}{24\theta^2}\right) = \frac{1}{3}\left(\theta^3 - \frac{1}{8}\right),$$

which is positive for $\theta > \frac{1}{2}$. Thus, it remains to consider deviations to $b \in \left(\frac{1}{2}, s(1)\right]$. Deviating to such b is equivalent to imitating some type in $\left(\frac{1}{2}, 1\right]$. This follows because the candidate equilibrium bidding function is strictly increasing in that range. The payoff from imitating $\hat{\theta} \in \left(\frac{1}{2}, 1\right]$ is

$$\hat{\theta}^2\theta - \hat{\theta}^2\left(\frac{2}{3}\hat{\theta} + \frac{1}{24\hat{\theta}^2}\right).$$

This is a strictly concave function in $\hat{\theta} \in \left(\frac{1}{2}, 1\right]$ and the FOC condition w.r.t. $\hat{\theta}$ is

$$2\hat{\theta}\theta - 2\hat{\theta}^2 = 0,$$

which implies $\hat{\theta} = \theta$. In all, no type θ would want to deviate from $s(\theta)$. ♦

5 Optional: The Universal Type Space

5.1 *Higher orders of belief.* We have considered a Bayesian game as a model of how a group of Bayesian players confront uncertainty. **Common prior assumption (CPA)** is useful in simplifying analysis, yet it makes several assumptions: (i) Θ is assumed to have a **product structure**; (ii) it is **common knowledge** that θ is drawn according to μ. That is to say, everyone knows μ, everyone knows that everyone else knows μ, etc. What if we relax the common prior assumption? That is to say, how should a group of Bayesian players **in general** behave when confronting uncertainty Θ?

If there is only one player, then the answer is simple. The Bayesian player comes up with a prior $\mu \in \Delta(\Theta)$ through introspection, then chooses some $s_1 \in S_1$ as to maximize $\int_{\theta\in\Theta} u_1(s_1, \theta)\, d\mu(\theta)$. The prior μ is trivially a common prior, since there is only one player.

However, in a game involving two players[28], the answer becomes far more complex. P1 is uncertain not only about state of the world Θ, but **also about P2's belief** over state of the world. P2's belief matters for P1's decision-making, since P1's utility depends on the pair (P1's action, P2's action) while P2's action depends on his belief. As a Bayesian must form a prior distribution over any relevant uncertainty, P1 should entertain not only a belief about state of the world, but also **a belief about P2's belief**, which is also unknown to P1.

To take a more concrete example, suppose there are two players Alice and Bob and the states of the world concern the weather tomorrow, $\Theta = \{\text{sunny}, \text{rain}\}$. Alice believes that there is a 60% chance that it is sunny tomorrow, 40% chance that it rains, so we say she has a **first-order belief** $\mu^{(1)}_{\text{Alice}} \in \Delta(\Theta)$ with $\mu^{(1)}_{\text{Alice}}(\text{sunny}) = 0.6$, $\mu^{(1)}_{\text{Alice}}(\text{rain}) = 0.4$. Now Alice needs to form a belief about Bob's belief regarding tomorrow's weather. Alice happens to know that Bob is a meteorologist who has access to more weather information than she does. In particular, Alice believes Bob's belief about weather tomorrow is **correlated with the actual weather** tomorrow. Either it is the case that tomorrow will be sunny and Bob believes today that it will be sunny tomorrow with probability 90%, or it is the case that tomorrow will rain and today Bob believes it will be sunny with probability 20%. Alice assigns 60-40 odds to these two cases. We say Alice has a **second-order belief** $\mu^{(2)}_{\text{Alice}} \in \Delta(\Theta \times \Delta(\Theta))$, where $\mu^{(2)}_{\text{Alice}}$ is supported on two points $(\text{sunny}, \mu^{(1)}_{\text{case 1}})$, $(\text{rain}, \mu^{(1)}_{\text{case 2}})$ with $\mu^{(2)}_{\text{Alice}}\left[\text{sunny}, \mu^{(1)}_{\text{case 1}}\right] = 0.6$, $\mu^{(2)}_{\text{Alice}}\left[\text{rain}, \mu^{(1)}_{\text{case 2}}\right] = 0.4$. Here $\mu^{(1)}_{\text{case 1}}$ and $\mu^{(1)}_{\text{case 2}}$ are elements of $\Delta(\Theta)$ and $\mu^{(1)}_{\text{case 1}}(\text{sunny}) = 0.9$ while $\mu^{(1)}_{\text{case 2}}(\text{sunny}) = 0.2$. We are not finished. Surely Bob, like Alice, also holds some second-order belief. Alice is uncertain about Bob's second-order belief, so she must form a **third-order belief**

$$\mu^{(3)}_{\text{Alice}} \in \Delta(\Theta \times \Delta(\Theta) \times \Delta(\Theta \times \Delta(\Theta)))$$

that is a joint distribution over (i) the weather tomorrow; (ii) Bob's first-order belief about the weather; (iii) Bob's second-order belief about the weather. Alice further needs a fourth-order belief, fifth-order belief, and so on.

We highlight the following features of the above example, which will be relevant to the subsequent theory on the universal type space:

- Alice entertains beliefs of order 1, 2, 3, ... about the state of the world, where kth-order belief is a joint distribution over state of the world, Bob's first order belief, Bob's second-order belief, ..., Bob's $(k-1)$th-order belief.
- Alice's second-order belief is **consistent** with her first-order belief, in the sense that whereas $\mu^{(1)}_{\text{Alice}}$ assigns probability of 60% to sunny weather tomorrow, $\mu^{(2)}_{\text{Alice}}$ marginalized to a distribution only over the weather also says there is a 60% chance that it is sunny tomorrow.
- There is **no common prior** over the weather and no signal structure is explicitly given.

Harsanyi (1967) first conjectured that for each specification of states of the world Θ, there corresponds an object now called the "**universal type space**"[29], say T. Points in the universal type space correspond to all "reasonable" hierarchies of first order belief, second order belief, third order belief, ... that a player could hold about Θ. Furthermore, there exists a "natural" homeomorphism

$$f : T \to \Delta(\Theta \times T)$$

so that each universal type t **encodes a joint belief** $f(t)$ over the state of the world and opponent's universal type. The universal type space is thus "universal" in the senses of (i) capturing all possible hierarchies of beliefs that might arise under some signal structure about Θ; (ii) putting an end to the seemingly infinite regress of having to resort to $(k+1)$th-order beliefs in order to model beliefs about kth-order beliefs, then having to discuss $(k+2)$th-order beliefs to describe beliefs about the $(k+1)$th-order beliefs just introduced, etc.

***5.2** Constructing the universal type space.* Mertens and Zamir (1985) first constructed the universal type space. Brandenburger and Dekel (1993) gave an alternative, simpler[30] construction, which we sketch here.

There are two players, i and j. The states of the world Θ is a Polish space (complete, separable metric space). For each Polish space Z, write $\Delta(Z)$ for the set of probability measures on Z's Borel σ-algebra. It is known that $\Delta(Z)$ is metrizable by the Prokhorov metric, which makes $\Delta(Z)$ a Polish space of its own right.

Iteratively, define $X_0 \equiv \Theta$, $X_1 \equiv \Theta \times \Delta(X_0)$, $X_2 \equiv \Theta \times \Delta(X_0) \times \Delta(X_1)$, etc. Each player has a first-order belief $\mu^{(1)}_i, \mu^{(1)}_j \in \Delta(X_0)$ that describes her belief about state of the world, a second-order belief $\mu^{(2)}_i, \mu^{(2)}_j \in \Delta(X_1)$ that describes

[28] All of this extends to games with 3 or more players, but with more cumbersome notations.

[29] Harsanyi initially called members of such space "attribute vectors". The word "type" only appeared in a later draft after Harsanyi discussed his research with Aumann and Maschler, who were also working on problems in information economics.

[30] Brandenburger and Dekel's construction was based on a slightly different set of assumptions than that of Mertens and Zamir. For instance, Mertens and Zamir assumed Θ is compact, but Brandenburger and Dekel required Θ to be a complete, separable metric space. Neither is strictly stronger.

her joint belief about state of the world and opponent's first order belief, and in general a kth-order belief $\mu_i^{(k)}, \mu_j^{(k)} \in \Delta(X_{k-1}) = \Delta(\Theta \times \Delta(X_0) \times ... \times \Delta(X_{k-2}))$ that describes her joint belief about state of the world, opponent's first order belief, ... and opponent's $(k-1)$-th order belief. Since X_0 is Polish, each X_k is Polish.

A **hierarchy of beliefs** is a sequence of beliefs of all orders, $(\mu_i^{(1)}, \mu_i^{(2)}, ...) \in \prod_{k=0}^{\infty} \Delta(X_k) \equiv T_0$. Note that there is a great deal of **redundancy** within the hierarchy. Indeed, as $\mu_i^{(k)}$ is a distribution over the first k elements of $\Theta, \Delta(X_0), \Delta(X_1), ...,$ each $\mu_i^{(k)}$ can be appropriately marginalized to obtain a distribution over the same domain as $\mu_i^{(k')}$ for any $1 \leq k' < k$. Call a hierarchy of beliefs **consistent** if each $\mu_i^{(k)}$ marginalized on all except the last dimension equals $\mu_i^{(k-1)}$ and write $T_1 \subset T_0$ for the subset of consistent hierarchies. Then, **Kolmogorov extension theorem** implies for each consistent hierarchy $(\mu_i^{(1)}, \mu_i^{(2)}, ...)$, there exists a measure $f(\mu_i^{(1)}, \mu_i^{(2)}, ...)$ over the infinite product $\Theta \times \left(\prod_{k=0}^{\infty} \Delta(X_k)\right)$ such that $f(\mu_i^{(1)}, \mu_i^{(2)}, ...)$ marginalized to $\Theta \times ... \times \Delta(X_{k-1})$ equals $\mu_i^{(k)}$ for each $k = 0, 1, 2, ...$ But $\Theta \times \left(\prod_{k=0}^{\infty} \Delta(X_k)\right)$ is in fact $\Theta \times T_0$, so that f associates each consistent hierarchy with a joint belief over state of the world and (possibly inconsistent) hierarchy of the opponent. Further, this association is **natural** in the sense that $f(\mu_i^{(1)}, \mu_i^{(2)}, ...)$ describes the same beliefs and higher order beliefs about Θ as the hierarchy $(\mu_i^{(1)}, \mu_i^{(2)}, ...)$. We may further verify the map $f : T_1 \to \Delta(\Theta \times T_0)$ is bijective, continuous, and has a continuous inverse, so that it is a homeomorphism.

To close the construction, define a sequence of decreasing subsets of T_1,

$$T_k \equiv \{t \in T_1 : [f(t)](\Theta \times T_{k-1}) = 1\}$$

That is, T_k is the subset of consistent types who put probability 1 on opponent's type being in the subset T_{k-1}. Let $T \equiv \bigcap_k T_k$, which is the class of types with "common knowledge of consistency": i "knows"[31] j's type is consistent, i "knows" that j "knows" i's type is consistent, etc. This is the **universal type space** over Θ. The map f can be restricted to the subset T to given a natural homeomorphism from T to $\Delta(\Theta \times T)$.

5.3 *Bayesian game as a belief-closed subset of the universal type space*. Here we discuss how the Bayesian game model relates to the universal type space.

Take a common prior Bayesian game $B = \langle N, (\Theta_i)_{i \in N}, (A_i)_{i \in N}, (u_i)_{i \in N}, \mu \rangle$ and suppose for simplicity there are two players, i and j. Each $\theta_i^* \in \Theta_i$ corresponds to a unique point in the universal type space T over Θ, which we write as $t(\theta_i^*) \in T$. To identify $t(\theta_i^*)$, note that player i of the type θ_i^* has a first-order belief $\mu_i^{(1)}[\theta_i^*] \in \Delta(\Theta)$, such that for $E_1 \subseteq \Theta$,

$$\mu_i^{(1)}[\theta_i^*](E_1) \equiv \mu(E_1 | \theta_i^*)$$

where $\mu(\cdot | \theta_i^*) \in \Delta(\Theta)$ is the conditional distribution on Θ derived from the common prior, given that $\theta_i = \theta_i^*$.

Furthermore, θ_i^* also leads to a second-order belief $\mu_i^{(2)}[\theta_i^*] \in \Delta(\Theta \times \Delta(\Theta))$, where for $E_1 \subseteq \Theta$, $E_2 \subseteq \Delta(\Theta)$,

$$\mu_i^{(2)}[\theta_i^*]\,(E_1 \times E_2) \equiv \mu\left(\left\{\hat{\theta} \in \Theta : \hat{\theta} \in E_1 \text{ and } \mu_j^{(1)}[\hat{\theta}_j] \in E_2\right\} \middle| \theta_i^*\right)$$

here $\mu_j^{(1)} : \Theta_j \to \Delta(\Theta)$ is defined analogously to $\mu_i^{(1)}$. One may similarly construct the entire hierarchy $t(\theta_i^*) = (\mu_i^{(k)}[\theta_i^*])_{k=1}^{\infty}$ and verify that it satisfies common knowledge of consistency. Hence, $t(\theta_i^*) \in T$. This justifies calling elements of Θ_i "types" of player i, for indeed they correspond to universal types over the states of the world.

The set of universal types present in the Bayesian game B, namely

$$T(B) \equiv \left\{t(\theta_k^*) : \theta_k^* \in \Theta_k, k \in \{i, j\}\right\}$$

is a **belief-closed subset** of T. That is, each $t \in T(B)$ satisfies $[f(t)]\,(\Theta \times T(B)) = 1$, putting probability 1 on the event that opponent is drawn from the set of universal types $T(B)$.

[31] More precisely, "knows" here means "puts probability 1 on".

(1) Subgame-perfect equilibrium; **(2)** Infinite-horizon games and one-shot deviation;

(3) Rubinstein-Stahl bargaining; **(4)** Introduction to repeated games; **(5)** Folk theorem for infinitely repeated games

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1 Subgame-Perfect Equilibrium

***1.1** Nash equilibrium in finite-horizon games.* Recall the definition of a finite-horizon extensive form game and the definition of a strategy in extensive form games from Section 1.

Definition 72. A (finite-horizon) **extensive form game** Γ consists of:

1. A (finite-depth) **tree** with vertices V and terminal vertices $Z \subseteq V$.
2. A (finite) collection of **players** $N = \{1, 2, ..., n\}$.
3. A **player function** $J : V \backslash Z \to N \cup \{c\}$. Denote $V_j = \{v : J(v) = j\}$ for each $j \in N \cup \{c\}$.
4. A set of available **moves** $M_{j,v}$ for each $j \in N$ and $v \in V_j$.
5. A **probability distribution** $f(\cdot|v)$ over v's children for each $v \in V_c$.
6. A (Bernoulli) **utility function** $u_j : Z \to \mathbb{R}$ for each $j \in N$.
7. An **information partition** $\mathcal{I}_j$ of V_j for each $j \in N$, whose elements are **information sets** $I_j \in \mathcal{I}_j$. It is required that $M_{j,v} = M_{j,v'}$ whenever $v, v' \in I_j$.

Definition 73. In an extensive form game, a **pure strategy** for player j is a function $s_j : \mathcal{I}_j \to \bigcup_{I_j \in \mathcal{I}_j} M_{I_j}$, so that $s_j(I_j) \in M_{I_j}$ for each $I_j \in \mathcal{I}_j$. Write S_j for the set of all pure strategies of player j. A **mixed strategy** for player j is an element $\sigma_j \in \Delta(S_j)$.

A pure strategy profile s induces a distribution over terminal vertices Z, which we write as $p(\cdot|s) \in \Delta(Z)$, where the randomness only comes from the moves of nature. Hence we may define, for each player i, $U_i : S \to \mathbb{R}$ where

$$U_i(s_i, s_{-i}) \equiv \mathbb{E}_{z \sim p(\cdot|s)} [u_i(z)] .$$

That is, the extensive game payoff to player i is defined as her expected utility from terminal vertices, according to her Bernoulli utility u_i and the distribution over terminal vertices induced by the strategy profile.

More generally, a mixed strategy profile σ also induces a distribution over terminal vertices Z, where now the randomness comes from both the moves of nature and the (independent) randomization of the players. We write $p(\cdot|\sigma) \in \Delta(Z)$ for the implied distribution over terminal vertices, and extend the domain of U_i to $\prod_{k \in N} \Delta(S_k)$, where

$$U_i(\sigma_i, \sigma_{-i}) \equiv \mathbb{E}_{z \sim p(\cdot|\sigma)} [u_i(z)] .$$

Note that we always assume that nature randomizes independently from the players.

A Nash equilibrium in extensive form game is defined in the natural way: a strategy profile where no player has a profitable unilateral deviation, where potential deviations are different extensive form game strategies.

Definition 74. A **Nash equilibrium** in an extensive form game is a strategy profile σ^* with $U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i' \in S_i$.

Example 75 (The ultimatum game[33])**.** Figure 7 shows the game tree of an **ultimatum game**, Γ. It models an interaction between players 1 and 2 who must split **two identical, indivisible items**. Player 1 proposes an allocation. Then, player 2 *A*ccepts or *R*ejects the allocation. If the allocation is accepted, it is implemented. If it is rejected, then neither player gets any of the good.

[32] Figure 13 is adapted from Osborne and Rubinstein (1994).

[33] The ultimatum game is an experimental economics game in which two parties interact anonymously and only once, so reciprocation is not an issue. The first player proposes how to divide a sum of money with the second party. If the second player rejects this division, neither gets anything.

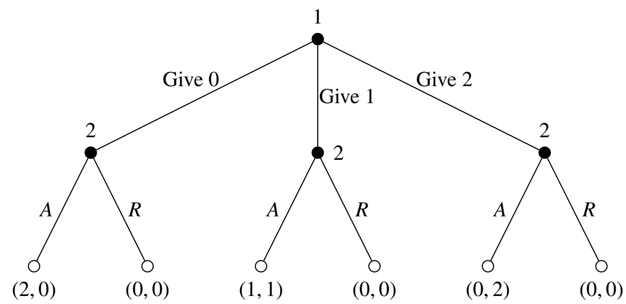

Figure 7: The ultimatum game in extensive form.

Player 1 moves at the root of the game tree. Her move set at the root is $\{0, 1, 2\}$, which correspond to giving 0, 1, 2 units of the good to player 2. Regardless of which action player 1 chooses, the game moves to a vertex where it is player 2's turn to play. His move set at each of his three decision vertices is $\{A, R\}$, corresponding to accepting and rejecting the proposed allocation.

The strategy profile $s_1^*(\varnothing) = 2$, $s_2^*(0) = s_2^*(1) = R$, $s_2^*(2) = A$ is a Nash equilibrium. Certainly player 2 has no profitable deviations since $U_2(s_1^*, s_2^*) = 2$, which is the highest he can hope to get in this game. As for player 1, she also has no profitable unilateral deviations, since offering 0 or 1 to player 2 leads to rejection and no change in her payoff. By the way, this is why we insist that a strategy in an extensive form game specifies what each player would do at each information set, even those information sets that are not reached when the game is played. What player 2 **would have done** if offered 0 or 1 is crucial in sustaining a Nash equilibrium in which player 1 offers 2. ♦

1.2 *Subgames and subgame-perfect equilibrium.* In some sense, the Nash equilibrium of Example 75 is artificially sustained by a **non-credible threat**. Player 2 threatens to reject the proposal if player 1 offers 1, despite the fact that he has no incentive to carry out the threat if player 1 really makes this offer. This threat does not harm player 2's payoff in the game Γ, since player 2's unoptimized decision vertex is never reached when the strategy profile (s_1^*, s_2^*) is played – it is "**off the equilibrium path**".

Whether or not strategy profiles like (s_1^*, s_2^*) make sense as predictions of the game's outcome depends on the availability of commitment devices. If at the start of the game player 2 could somehow make it impossible for himself to accept the even-split offer, then this Nash equilibrium is a reasonable prediction. In the absence of such commitment devices, however, we should seek out a refinement of Nash equilibrium in extensive form games to rule out such non-credible threats.

We begin with the definition of a subgame.

Definition 76 (Subgame)**.** In a finite-horizon extensive form game Γ, any vertex $x \in V \backslash Z$ such that every information set is either entirely contained in the subtree starting at x or entirely outside of it defines a **subgame**, $\Gamma(x)$. This subgame is an extensive form game inherits the payoffs, moves, and information structure of the original game Γ in the natural way.

Example 77 (The ultimatum game)**.** The ultimatum game in Example 75 has 4 subgames: $\Gamma(\varnothing)$ (which is just Γ), as well as $\Gamma(0), \Gamma(1), \Gamma(2)$. We sometimes call $\Gamma(0), \Gamma(1), \Gamma(2)$ the **proper subgames**. ♦

Definition 78 (Subgame-perfect equilibrium)**.** A strategy profile σ^* of Γ is called a **subgame-perfect equilibrium (SPE)** if for every subgame $\Gamma(x)$, σ^* restricted to $\Gamma(x)$ forms a Nash equilibrium in $\Gamma(x)$.

Note that we can rewrite mixed strategies as behavioral strategies in games with perfect recall. Then, for each player, the restriction of the strategies to a subgame is just the collection of the behavioral strategies corresponding/relevant to information sets in the subgame.

$\Gamma(\varnothing) = \Gamma$ is always a subgame since the root of the game tree is always in a singleton information set. Therefore, every SPE is an NE, but not conversely.

Example 79 (The ultimatum game)**.** The NE (s_1^*, s_2^*) from Example 75 is not an SPE, since (s_1^*, s_2^*) restricted to the subgame $\Gamma(1)$ is not an NE. However, the following is an SPE: $s_1^\circ(\varnothing) = 1$, $s_2^\circ(0) = R$, $s_2^\circ(1) = s_2^\circ(2) = A$. It is easy to see that restricting (s_1°, s_2°) to each of the subgames $\Gamma(0), \Gamma(1), \Gamma(2)$ forms an NE. Furthermore, (s_1°, s_2°) is a NE in $\Gamma(\varnothing) = \Gamma$. Player 1 gets $U_1(s_1^\circ, s_2^\circ) = 1$ under this strategy profile, while offering 0 leads to rejection and a payoff of 0, offering 2 leads to acceptance but again a payoff of 0. For player 2, changing $s_2^\circ(0)$ and $s_2^\circ(2)$ do not change payoff in Γ, since these two vertices are never reached. Changing $s_2^\circ(1)$ from A to R hurts payoff. ♦

1.3 Backward induction. **Backward induction** is an algorithm for finding SPE in a finite-horizon extensive form game of perfect information. The idea is to successively replace subgames with terminal vertices corresponding to SPE payoffs of the deleted subgames.

Start with a non-terminal vertex **furthest away** from the root of the game, say v. Since we have picked the deepest non-terminal vertex, all of $J(v)$'s moves at this vertex must lead to terminal vertices. Choose one of $J(v)$'s moves, m^*, that maximizes her payoff in $\Gamma(v)$, then replace the subgame $\Gamma(v)$ with the terminal vertex corresponding to m^*. Repeat this procedure, working backwards from the vertices further away from the root of the game. Eventually, the game tree will be reduced to a single terminal vertex, whose payoff will be an SPE payoff of the extensive form game, while the moves chosen throughout the deletion process will form a SPE strategy profile.

Example 80. Figure 8 through 12 display the process of backward induction.

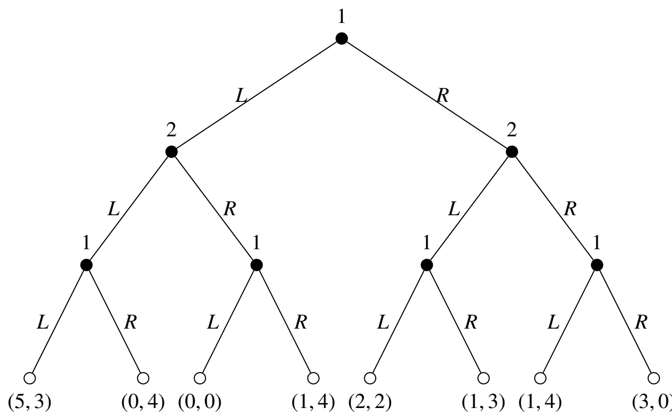

Figure 8: An extensive form game.

Backward induction replaces subgames with terminal nodes associated with the SPE payoffs in those subgames. Here is the resulting game tree after one step of backward induction.

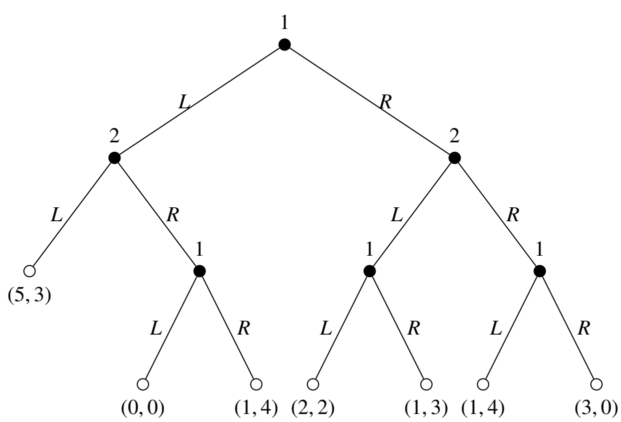

Figure 9: The resulting game tree after one step of backward induction.

Proceed similarly, after eliminating all nodes at depth 3 in the original tree, the results look as follows:

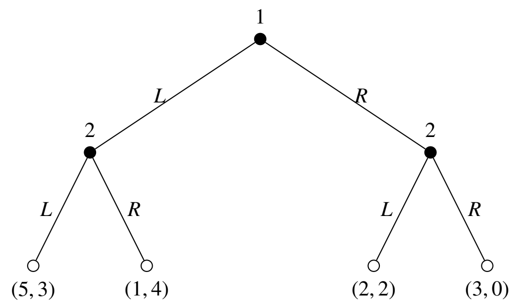

Figure 10: Backward induction in progress. All nodes at depth 3 in the original tree have been eliminated.

We can continue:

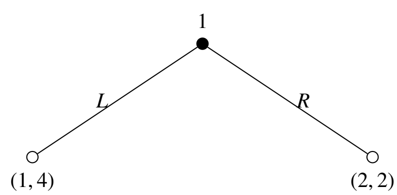

Figure 11: Backward induction in progress. Only nodes with depth 1 remain.

And continue again:

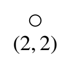

Figure 12: Backward induction finds the unique SPE payoff in this game.

This completes backward induction, and we have found the unique SPE of the original game, s^*: $s_1^*(\varnothing) = R$, $s_1^*(L, L) = L$, $s_1^*(L, R) = R$, $s_1^*(R, L) = L$, $s_1^*(R, R) = R$, $s_2^*(L) = R$, $s_2^*(R) = L$. ♦

If $u_i(z) \neq u_i(z')$ for every i and $z \neq z'$, then backward induction finds the unique SPE of the extensive form game. Otherwise, the game may have multiple SPEs and backward induction may involve choosing between several indifferent moves. Depending on the moves chosen, backward induction may lead to different SPEs.

Example 81 (From MWG[34]**).** Consider a game in which the following simultaneous-move game is played twice. The players observe the actions chosen in the first period before they play in the second period. What are the pure strategy SPEs of this game?

	b_1	b_2	b_3
a_1	10, 10	2, 12	0, 13
a_2	12, 2	5, 5	0, 0
a_3	13, 0	0, 0	1, 1

Solution:

The pure strategy Nash equilibria of the one-shot game are (a_2, b_2) and (a_3, b_3). Thus any pure strategy SPE involves playing either of these in the second period. We conjecture the following four classes of SPE:

1. Players plays (a_i, b_i) in both periods, $i \in \{2, 3\}$.

[34] This is a prelude to repeated games, which are discussed later.

2. Players plays (a_i, b_i) in the first period and (a_j, b_j) in the second period, $i, j \in \{2, 3\}$ and $i \neq j$.

3. Player 1's strategy: play a_i, $i \in \{1, 2, 3\}$ in period 1; play a_2 in period 2 if player 2 played b_1 in period 1, otherwise play a_3.

   Player 2's strategy: play b_1 in period 1; play b_2 in period 2 if player 2 played a_i in period 1, otherwise play b_3.

4. Player 2's strategy: play b_i, $i \in \{1, 2, 3\}$ in period 1; play b_2 in period 2 if player 2 played a_1 in period 1, otherwise play b_3.

   Player 1's strategy: play a_1 in period 1; play a_2 in period 2 if player 2 played b_i in period 1, otherwise play a_3.

Classes 1 and 2 are easy to check. To see that classes 3 and 4 are indeed SPEs, note that by deviating a player loses 4 in the second period and no player can gain more than 3 in any of the described strategy profiles. Equilibrium classes 3 and 4 are implemented through 'punishment for deviations'. ♦

2 Infinite-Horizon Games and One-Shot Deviation

2.1 *Infinite-horizon games.* So far, we have only dealt with finite-horizon games. These games are represented by finite-depth game trees and must end within M turns for some finite M. But games such as the Rubinstein-Stahl bargaining are not finite-horizon, for players could reject each other's offers forever. We modify Definition 82 to accommodate such infinite-horizon games. For simplicity, we assume the game has perfect information and no chance moves.

Definition 82. An **extensive form game** with **perfect information** and **no chance moves** Γ consists of:

1. A (possibly infinite-depth) **tree** with vertices V and terminal vertices $Z \subseteq V$.

2. A (finite) collection of **players** $N = \{1, 2, ..., n\}$.

3. A **player function** $J : V \backslash Z \to N$. Denote $V_j = \{v : J(v) = j\}$ for each $j \in N$.

4. A set of available **moves** $M_{j,v}$ for each $j \in N$ and $v \in V_j$. Each move in $M_{j,v}$ is associated with a unique child of v in the tree.

5. A set of **infinite histories** H^∞, where each $h^\infty \in H^\infty$ represents an infinite-length path $(v_0, v_1, \dots)$ in the tree.

6. A (Bernoulli) **utility function** $u_j : Z \cup H^\infty \to \mathbb{R}$ for each $j \in N$.

When an infinite-horizon game is played, it might end at a terminal vertex (such as when one player accepts the other's offer in the bargaining game), or it might **never reach a terminal vertex** (such as when both players use a strategy involving never accepting any offer in the bargaining game). Therefore, each player must have a preference not only over the set of terminal vertices, but also **over the set of infinite histories**. In the bargaining game, for instance, it is specified that $u_j(h^\infty) = 0$ for any $h^\infty \in H^\infty$, $j = 1, 2$, that is to say every infinite history in the game tree (i.e., never reaching an agreement) gives 0 utility to each player.

Many definitions from finite-horizon extensive form games directly translate into the infinite-horizon setting. For instance, any nonterminal vertex x in the perfect-information infinite-horizon game defines a subgame $\Gamma(x)$. NE is defined in the obvious way, taking into account distribution over both terminal vertices and infinite histories induced by a strategy profile. SPE is still defined as those strategy profiles that form an NE when restricted to each of Γ's (possibly infinitely many) subgames.

2.2 *One-shot deviation principle*. It is often difficult to verify directly from definition whether a given strategy profile forms an SPE in an infinite-horizon game. Indeed, given an SPE candidate s^* of game Γ, we would have to consider each subgame $\Gamma(x)$, which is potentially an infinite-horizon extensive form game of its own right, and ask whether player i can improve her payoff in $\Gamma(x)$ by choosing a different extensive form game strategy s_i', modifying some or all of her choices at various vertices in V_i relative to s_i^*. This is not an easy task since i's set of strategies in $\Gamma(x)$ is a very rich set. The one-shot deviation principle says for extensive form games satisfying certain regularity conditions, we need only check that i does not have a profitable deviation amongst **a very restricted set** of strategies in each subgame $\Gamma(x)$, namely those that differ from s_i^* only at x.

Definition 83 (Continuous at infinity)**.** Γ is **continuous at infinity** if for all $\varepsilon > 0$, there exists an integer T such that for every player i and any two infinite histories $h^\infty, \tilde{h}^\infty \in H^\infty$ that share the first T nodes, $\left|u_i(h^\infty) - u_i(\tilde{h}^\infty)\right| < \varepsilon$.

Theorem 84 (One-shot deviation principle)**.** *If Γ is continuous at infinity, then a strategy profile s^* is an SPE of Γ if and only if for every player i, every $x \in V_i$ and every strategy s'_i such that $s^*_i(v) = s'_i(v)$ at every $v \neq x$,*

$$U_i(s^*_i, s^*_{-i}|x) \geq U_i(s'_i, s^*_{-i}|x)$$

where $U_i(\cdot|x)$ denotes the payoff of i in subgame $\Gamma(x)$.

Example 85 (A game not continuous at infinity)**.** Consider an infinite-horizon one-player game, where the player chooses *Y*es or *N*o in each stage. She gets payoff 1 if she chooses Y forever, and 0 if she ever chooses N. Always N is not SPE, yet it satisfies the condition of the one-shot deviation principle. This is because the game is not continuous at infinity. For any integer T, consider two infinite histories that share the first T nodes, $h^\infty = \{\text{Always } Y\}$, $\tilde{h}^\infty = \{\text{Always } Y \text{ except at the } (T+1)\text{th node}\}$. We have $\left|u_i(h^\infty) - u_i(\tilde{h}^\infty)\right| = 1$. This example shows that continuity at infinity is crucial for one-shot deviation principle to hold. ♦

Continuity at infinity is satisfied by all finite-horizon extensive form games, as well as all infinite-horizon games studied in lecture, including bargaining and repeated games. Under this condition, to verify whether s^* is an SPE, we only need to examine each subgame $\Gamma(x)$ and consider whether player $J(x)$ can improve her payoff in $\Gamma(x)$ by changing her move only at x (a "**one-shot deviation**").

3 Rubinstein-Stahl Bargaining

3.1 *Bargaining as an extensive form game*. The **Rubinstein-Stahl bargaining game**, or simply "**bargaining game**"[35] for short, is an important example of infinite-horizon, perfect-information extensive form game. It is comparable to the ultimatum game from Example 75, but with two important differences: (i) the game is **infinite-horizon**, so that first rejection does not end the game. Instead, players alternate in making offers; (ii) the good that players bargain over is assumed **infinitely divisible**, so that any allocation of the form $(x, 1 - x)$ for $x \in [0, 1]$ is feasible.

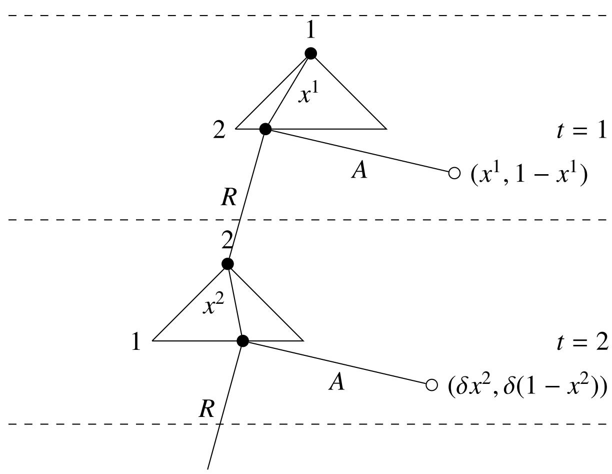

Figure 13: Part of the bargaining game tree, showing only some of the branches in the first two periods. The root ∅ has (uncountably) infinitely many children of the form $(x^1, 1 - x^1)$ for $x^1 \in [0, 1]$. At each such child, player 2 may play R or A. Playing A leads to a terminal node with payoffs $(x^1, 1 - x^1)$, while playing R continues the game with player 2 to make the next offer.

Let's think about what a strategy in the bargaining game looks like. Figure 13 shows a sketch of the bargaining game tree. Player 1's strategy specifies $s_1(\varnothing)$, that is to say what player 1 will offer at the start of the game. For each $x^1 \in [0, 1]$, player 2's strategy specifies $s_2((x^1, 1 - x^1)) \in \{A, R\}$, that is whether he accepts or rejects a period 1 offer of $(x^1, 1 - x^1)$. In addition, player 2's strategy must also specify $s_2((x^1, 1 - x^1), R)$ for each $x^1 \in [0, 1]$, that is what he offers in period $t = 2$ if he rejected player 1's offer in $t = 1$.[36] This offer could in principle could depend on what player 1 offered in period $t = 1$. Now for every $x^1, x^2 \in [0, 1]$, player 1's strategy must specify

[35] Not to be confused with axiomatic Nash bargaining, which we will study in Jerry's part in 2010b.

[36] Remember, a strategy for j is a complete contingency plan that specifies a valid move at any vertex in the game tree where it is j's turn to play, even those vertices that would never be reached due to how j plays in previous rounds. Even if player 2's strategy specifies accepting every offer from player 1 in $t = 1$, player 2 still needs to specify what he would do after a history of the form $((x^1, 1 - x^1), R)$ for each $x^1 \in [0, 1]$.

$s_1((x^1, 1-x^1), R, (x^2, 1-x^2)) \in \{A, R\}$, which could in principle depend on what she herself offered in period $t = 1$, as well as what player 2 offered in the current period, $(x^2, 1-x^2)$.

3.2 *Asymmetric bargaining power*. Here is a modified version of the bargaining game that introduces **asymmetric bargaining power** between the two players.

Example 86. P1 gets to make offers in periods $3k+1$ and $3k+2$, while P2 gets to make offers in periods $3k+3$. As in the usual bargaining game, reaching an agreement of $(x, 1-x)$ in period t yields the payoff profile $(\delta^{t-1} \cdot x, \delta^{t-1} \cdot (1-x))$. If the players never reach an agreement, then payoffs are $(0, 0)$.

Consider the following strategy profile: whenever P1 makes an offer in period $3k+1$, she offers $\left(\frac{1+\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right)$. Whenever P1 makes an offer in period $3k+2$, she offers $\left(\frac{1+\delta^2}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}\right)$. Whenever P2 makes an offer, he offers $\left(\frac{\delta+\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}\right)$. Whenever P2 responds to an offer in period $3k+1$, he accepts if and only if he gets at least $\frac{\delta^2}{1+\delta+\delta^2}$. Whenever P2 responds to an offer in period $3k+2$, he accepts if and only if he gets at least $\frac{\delta}{1+\delta+\delta^2}$. Whenever P1 responds to an offer, she accepts if and only if she gets at least $\frac{\delta+\delta^2}{1+\delta+\delta^2}$. You may verify that this verbal description indeed defines a strategy profile that plays a valid move at every non-terminal node of the bargaining game tree.

We use the **one-shot deviation principle** to verify that this strategy profile is SPE. By the principle, we need only ensure that in each subgame, the player to move at the root of the subgame cannot gain by changing her move only at the root. Subgames of this bargaining game may be classified into **six families**:

1. Subgame starting with P1 making an offer in period $3k+1$.
2. Subgame starting with P1 making an offer in period $3k+2$.
3. Subgame starting with P2 making an offer in period $3k+3$.
4. Subgame starting with P2 responding to an offer $(x, 1-x)$ in period $3k+1$.
5. Subgame starting with P2 responding to an offer $(x, 1-x)$ in period $3k+2$.
6. Subgame starting with P1 responding to an offer $(x, 1-x)$ in period $3k+3$.

We consider these six families one by one, showing in no subgame is there a profitable one-shot deviation. By the one-shot deviation principle, this shows the strategy profile is an SPE.

1. **Subgame starting with P1 making an offer in period $3k+1$.**

   Not deviating gives P1 $\delta^{3k} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. Offering P2 more than $\frac{\delta^2}{1+\delta+\delta^2}$ leads to acceptance but yields strictly less utility to P1. Offering P2 less than $\frac{\delta^2}{1+\delta+\delta^2}$ leads to rejection. In the next period, P1 will offer herself $\frac{1+\delta^2}{1+\delta+\delta^2}$, which P2 will accept. Therefore, this deviation gives P1 utility $\delta^{3k+1} \cdot \frac{1+\delta^2}{1+\delta+\delta^2} < \delta^{3k} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. So we see P1 has no profitable one-shot deviation at the start of this subgame.

2. **Subgame starting with P1 making an offer in period $3k+2$.**

   Not deviating gives P1 $\delta^{3k+1} \cdot \frac{1+\delta^2}{1+\delta+\delta^2}$. Offering P2 more than $\frac{\delta}{1+\delta+\delta^2}$ leads to acceptance but yields strictly less utility to P1. Offering P2 less than $\frac{\delta}{1+\delta+\delta^2}$ leads to rejection. In the next period, P2 will offer P1 $\frac{\delta+\delta^2}{1+\delta+\delta^2}$, which P1 will accept. Therefore, this deviation gives P1 utility $\delta^{3k+2} \cdot \frac{\delta+\delta^2}{1+\delta+\delta^2} < \delta^{3k+1} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. So we see P1 has no profitable one-shot deviation at the start of this subgame.

3. **Subgame starting with P2 making an offer in period $3k+3$.**

   Not deviating gives P2 $\delta^{3k+2} \cdot \frac{1}{1+\delta+\delta^2}$. Offering P1 more than $\frac{\delta+\delta^2}{1+\delta+\delta^2}$ leads to acceptance but yields strictly less utility to P2. Offering P1 less than $\frac{\delta+\delta^2}{1+\delta+\delta^2}$ leads to rejection. In the next period, P1 will offer P2 $\frac{\delta^2}{1+\delta+\delta^2}$, which P2 will accept. Therefore, this deviation gives P2 utility $\delta^{3k+3} \cdot \frac{\delta^2}{1+\delta+\delta^2} < \delta^{3k+2} \cdot \frac{1}{1+\delta+\delta^2}$. So we see P2 has no profitable one-shot deviation at the start of this subgame.

4. **Subgame starting with P2 responding to an offer $(x, 1-x)$ in period $3k+1$.**

   If $1-x < \frac{\delta^2}{1+\delta+\delta^2}$, the strategy for P2 prescribes rejection. In the next period, P1 will offer P2 $\frac{\delta}{1+\delta+\delta^2}$ which P2 will accept, giving P2 a utility of $\delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2}$. On the other hand, the deviation of accepting $1-x$ in the current period gives utility $\delta^{3k} \cdot (1-x) < \delta^{3k} \cdot \frac{\delta^2}{1+\delta+\delta^2} = \delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2}$. So P2 has no profitable one-shot deviation.

   If $1-x \geq \frac{\delta^2}{1+\delta+\delta^2}$, the strategy for P2 prescribes acceptance, giving P2 a utility of $\delta^{3k} \cdot (1-x) \geq \delta^{3k} \cdot \frac{\delta^2}{1+\delta+\delta^2}$. If P2 rejects instead, then in the next period P1 will offer P2 $\frac{\delta}{1+\delta+\delta^2}$ which P2 will accept, giving P2 a utility of $\delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2} \leq \delta^{3k} \cdot \frac{\delta^2}{1+\delta+\delta^2}$. So P2 has no profitable one-shot deviation.

5. **Subgame starting with P2 responding to an offer** $(x, 1-x)$ **in period** $3k+2$.

   If $1-x < \frac{\delta}{1+\delta+\delta^2}$, the strategy for P2 prescribes rejection. In the next period, P2 will offer himself $\frac{1}{1+\delta+\delta^2}$, which P1 will accept, giving P2 a utility of $\delta^{3k+2} \cdot \frac{1}{1+\delta+\delta^2}$. On the other hand, the deviation of accepting $1-x$ in the current period gives P2 utility $\delta^{3k+1} \cdot (1-x) < \delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2} = \delta^{3k+2} \cdot \frac{1}{1+\delta+\delta^2}$. So P2 has no profitable one-shot deviation.

   If $1-x \geq \frac{\delta}{1+\delta+\delta^2}$, the strategy for P2 prescribes acceptance, giving P2 utility of $\delta^{3k+1} \cdot (1-x) \geq \delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2}$. If P2 rejects instead, then in the next period P2 will offer himself $\frac{1}{1+\delta+\delta^2}$, which P1 will accept, giving P2 a utility of $\delta^{3k+2} \cdot \frac{1}{1+\delta+\delta^2} = \delta^{3k+1} \cdot \frac{\delta}{1+\delta+\delta^2} \leq \delta^{3k+1} \cdot (1-x)$. So P2 has no profitable one-shot deviation.

6. **Subgame starting with P1 responding to an offer** $(x, 1-x)$ **in period** $3k+3$.

   If $x < \frac{\delta+\delta^2}{1+\delta+\delta^2}$, the strategy for P1 prescribes rejection. In the next period, P1 will offer herself $\frac{1+\delta}{1+\delta+\delta^2}$, which P2 will accept, giving P1 a utility of $\delta^{3k+3} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. On the other hand, the deviation of accepting x in the current period gives P1 utility $\delta^{3k+2} \cdot (x) < \delta^{3k+2} \cdot \frac{\delta+\delta^2}{1+\delta+\delta^2} = \delta^{3k+3} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. So P1 has no profitable one-shot deviation.

   If $x \geq \frac{\delta+\delta^2}{1+\delta+\delta^2}$, the strategy for P1 prescribes acceptance, giving P1 utility of $\delta^{3k+2} \cdot (x) \geq \delta^{3k+3} \cdot \frac{1+\delta}{1+\delta+\delta^2}$. If P1 rejects instead, then in the next period P1 will offer herself $\frac{1+\delta}{1+\delta+\delta^2}$, which P2 will accept, giving P1 a utility of $\delta^{3k+3} \cdot \frac{1+\delta}{1+\delta+\delta^2} \leq \delta^{3k+2} \cdot (x)$. So P1 has no profitable one-shot deviation.

Along the equilibrium path, P1 offers $\left(\frac{1+\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right)$ in $t=1$ and P2 accepts. This is a better outcome for P1 than in the symmetric bargaining game where P1 gets $\frac{1}{1+\delta}$. P1's SPE payoff improves when she has more bargaining power.

One can also show that the payoffs of this SPE are the unique SPE payoffs of the game. The proof idea is to consider three bargaining games: Γ_1, Γ_2 and Γ_3. Γ_1 is the bargaining game exhibited in the statement of the problem. Γ_2 is the bargaining game, where P1 makes an offer in period one, P2 makes an offer in period two and if both reject the bargaining game Γ_1 is played. Γ_3 is the bargaining game where P2 makes the first offer in period one and if P1 rejects the bargaining game Γ_1 is played.

Define the minimal, maximal SPE payoffs for both players in all three bargaining games and use an analysis similar to lecture to find necessary inequalities these minimal, maximal SPE payoffs have to satisfy. Ultimately, one arrives at 'enough' inequalities so that a combination of them gives a tight characterization showing that the minimal and maximal SPE payoffs for both players are equal to the payoffs in the SPE considered in this example. ♦

4 Introduction to Repeated Games

***4.1** What is a repeated game?* Many of the normal form and extensive form games studied so far can be viewed as **models of one-time encounters**. After players finish playing Rubinstein-Stahl bargaining or high-bid auction, they part ways and never interact again. In many economic situations, however, a group of players may **play the same game again and again** over a long period of time. For instance, a customer might approach a printing shop every month with a major printing job. While the printing shop has an incentive to shirk and produce low-quality output in a one-shot version of this interaction, in a long-run relationship the shop might never shirk as to avoid losing the customer in the future. In general, repeated games study what outcomes can arise in such repeated interactions.

Formally speaking, repeated games (with perfect monitoring[37]) form an important class of examples within extensive form games with finite- or infinite-horizon, depending on the length of repetition.

Definition 87 (Finitely repeated game)**.** For a normal form game $G = \langle N, (A_k)_{k\in N}, (u_k)_{k\in N}\rangle$ and a positive integer T, denote by $G(T)$ the extensive form game where G is played in every period for T periods and players observe the action profiles from all previous periods. G is called the **stage game** and $G(T)$ the **T-times repeated game**. **Terminal vertices** of $G(T)$ are of the form $h^T = (a^1, a^2, ..., a^T) \in A^T$ and **payoff** to player i at such a terminal vertex is

$$U_i(h^T) \equiv \sum_{t=1}^{T} u_i(a^t).$$

A **pure strategy** for player i maps each non-terminal history of action profiles to a stage game action,

$$s_i : \bigcup_{k=0}^{T-1} A^k \to A_i.$$

[37] That is to say, actions taken in previous periods are common knowledge. There exists a rich literature on repeated games with coarser monitoring structures – for instance, all players observe an imperfect public signal of each period's action profile, or each player privately observes such a signal – and folk theorems in these generalized settings (Fudenberg, Levine, and Maskin, 1994; Kandori and Matsushima, 1998; Sugaya, Forthcoming).

Definition 88 (Infinitely repeated game)**.** For a normal form game $G = \langle N, (A_k)_{k\in N}, (u_k)_{k\in N}\rangle$ and $\delta \in [0, 1)$, denote by $G^\delta(\infty)$ the extensive form game where G is played in every period for infinitely many periods and players act like exponential discounters with discount factor δ. $G^\delta(\infty)$ is called the **infinitely repeated game with discount factor** δ. An **infinite history** of the form $h^\infty = (a^1, a^2, ...,) \in A^\infty$ gives player i the **payoff**

$$U_i(h^\infty) \equiv \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t).$$

A **pure strategy** for player i maps each finite history of action profiles to a stage game action,

$$s_i : \bigcup_{k=0}^{\infty} A^k \to A_i.$$

A strategy for player i in $G(T)$ or $G^\delta(\infty)$ must specify a valid action of the stage game G after **any** non-terminal history $(a^1, ..., a^k) \in A^k$, including those histories that would never be reached under i's strategy. For example, even if P1's strategy in repeated prisoner's dilemma is to always play D, she still needs to specify $s_1((C, C), (C, C))$, that is what she will play in period 3 if both players cooperated in the first two periods.

As defined above, our treatment of repeated games focuses on the the simplest case where payoffs in period t are **independent of actions taken in all previous periods**. This rules out, for instance, investment games where players choose a level of contribution every period and the utility in period t depends on the sum of all accumulated capital up to period t.

When discussing repeated games, we are often interested in the "average" stage game payoff under a repeated game strategy profile. The following definitions are just **normalizations**: they ensure that the (finite or infinite) constant action profile $(a, a, \dots)$ leads to an average payoff of $u_i(a)$.

Definition 89 (Average payoff)**.** In $G(T)$, the **average payoff** to i at a terminal vertex $h^T = (a^1, a^2, ..., a^T) \in A^T$ is

$$\overline{U}_i(h^T) \equiv \frac{1}{T} \sum_{t=1}^{T} u_i(a^t).$$

In $G^\delta(\infty)$, the (**discounted**) **average payoff** to i at the infinite history $h^\infty = (a^1, a^2, ...,) \in A^\infty$ is

$$\overline{U}_i(h^\infty) \equiv (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t).$$

***4.2** Some immediate results.* The first result is immediate from backward induction.

Proposition 90. *If G has a unique NE, then for any finite T, the repeated game $G(T)$ has a unique SPE. In this SPE, players play the unique stage game NE after every non-terminal history.*

Proof. Let σ^* be an SPE of $G(T)$. For any history h^{T-1} of length $T-1$, $\sigma^*(h^{T-1})$ must be the unique NE of G. Else, some player must have a strictly profitable deviation in the last period T. So we deduce σ^* plays the unique NE of G in period T regardless of what happened in previous periods.

But this means $\sigma^*(h^{T-2})$ must also be the unique NE of G for any history h^{T-2} of length $T-2$. otherwise, consider the subgame starting at h^{T-2}. If $\sigma^*(h^{T-2})$ does not form an NE, some player i can improve her payoff in the current period by changing her action in period $T-1$, and furthermore this change does not affect her payoff in future periods, since we have argued the unique NE of G will be played in period T regardless of what happened earlier in the repeated game. So we have found a strictly profitable deviation for i in the subgame, contradicting the fact that σ^* is an SPE.

Hence, we have shown σ^* plays the unique NE of G in the last two periods of $G(T)$, regardless of what happened earlier. Continuing this argument shows the unique NE of G is played after any non-terminal history. □

The second result requires some additional definitions.

Definition 91 (Feasible payoffs)**.** Given a normal form game $G = \langle N, (A_k)_{k\in N}, (u_k)_{k\in N}\rangle$, the set of **feasible payoffs** is defined as $\text{co}(\{u(a) : a \in A\}) \subseteq \mathbb{R}^n$, where $\text{co}(\cdot)$ is the convex hull operator.

These are the payoffs that can be obtained if players use a **public randomization device** to correlate their actions. Specifically, as every $v \in \text{co}(\{u(a) : a \in A\})$ can be written as a weighted average $v = \sum_{\ell=1}^{r} p_\ell u(a^{(\ell)})$ where $p_\ell \geq 0$, $\sum_{\ell=1}^{r} p_\ell = 1$ and $a^{(\ell)} \in A$ for each ℓ, one can construct a correlated strategy profile where all players observe a public random variable that realizes to ℓ with probability p_ℓ, then player i plays $a_i^{(\ell)}$ upon observing ℓ. The expected payoff profile under this correlated strategy profile is v.

This public randomization device will be used in the construction of the equilibrium in two ways:

1. To realize specific feasible payoffs in certain periods as described above (on or off equilibrium path). It follows from the optimization property of the equilibrium that agents will follow the prescriptions of the public randomization device (i.e., the incentives are given by the continuation play encoded in the equilibrium strategies).

2. To construct SPEs which have as payoffs a mixture of the payoffs from two (or more) SPEs. An illustrative example is as follows: imagine that we have two SPE profiles, $\sigma^{(1)}$ and $\sigma^{(2)}$, which give SPE payoff profiles $U(\sigma^{(1)})$ and $U(\sigma^{(2)})$. Then, given $\lambda \in (0, 1)$, agents can achieve the SPE payoff profile $\lambda U(\sigma^{(1)}) + (1 - \lambda)U(\sigma^{(2)})$ by using the public randomization device and replicating a toss of coin with heads probability of λ at time $t = 0$ before play starts: if heads up, then play SPE $\sigma^{(1)}$, otherwise play $\sigma^{(2)}$. This is an SPE, albeit now constructed by public randomization, because no matter what the public outcome of the coin is, the agents will follow its prescription due to the SPE property of $\sigma^{(1)}$ and $\sigma^{(2)}$.

Definition 92 (Minimax payoff)**.** In a normal form game $G = \langle N, (A_k)_{k\in N}, (u_k)_{k\in N}\rangle$, player i's **minimax payoff** is defined as

$$\underline{v}_i \equiv \min_{\alpha_{-i}\in\Delta(A_{-i})} \max_{a_i\in A_i} u_i(a_i, \alpha_{-i}).$$

Definition 93 (Individually rational)**.** In a normal form game $G = \langle N, (A_k)_{k\in N}, (u_k)_{k\in N}\rangle$, call a payoff profile $v \in \mathbb{R}^n$ **individually rational (IR)** if $v_i \geq \underline{v}_i$ for every $i \in N$. Call v **strictly individually rational** if $v_i > \underline{v}_i$ for every $i \in N$.

Two technical remarks which you can skip:

1. The outer minimization in minimax payoff is across the set of **correlated** strategy profiles of $-i$. As demonstrated in the coordination game with an eavesdropper (Example 32), the correlated minimax payoff of a player could be **strictly lower** than her independent minimax payoff (when opponents in $-i$ play independently mixed actions). This distinction is not very important for this course, as we will almost always consider two-player stage games when studying repeated games, so that the set of "correlated" strategy profiles of $-i$ is just the set of mixed strategies of $-i$.

2. We claimed to have described repeated games with **perfect monitoring** in Definitions 87 and 88, but the monitoring structure as written was less than perfect. Players only observe past actions and **cannot always detect deviations from mixed strategies or correlated strategies**, so in particular they do not know for sure if everyone is faithfully playing a mixed or correlated minimax strategy profile against i.[38] To remedy this problem, we can assume that every coalition (including singleton coalitions) observes a correlating signal at the start of every period, which they use to implement correlated strategies and mixed strategies. Furthermore, the realizations of such correlating signals become publicly known the end of each period, so that even correlated strategies and mixed strategies are "observable". This remark is again not very important for this course, for the minimax action profile turns out to be pure in most stage games we examine. In addition, Fudenberg and Maskin (1986) showed that their folk theorem continues to hold, albeit with a modified proof, even when players only observe past actions and not the realizations of past correlating devices.

Proposition 94. *Suppose σ^* is a Nash equilibrium for $G(T)$ or $G^\delta(\infty)$. Then the average payoff profile associated with σ^* is feasible and IR for the stage game G.*

Proof. Evidently, the payoff profile in every period of the repeated game must be in $\text{co}(\{u(a) : a \in A\})$. In $G(T)$, the average payoff profile under σ^* is the simple average of T such points, while in $G^\delta(\infty)$ it is a weighted average of countably many such points, so in both cases the average payoff profile must still be in $\text{co}(\{u(a) : a \in A\})$ by the convexity of this set.

Suppose now player i's average payoff is strictly less than $\underline{v}_i$. Then consider a new repeated game strategy σ_i' for i, where $\sigma_i'(h)$ best responds to the (possibly correlated) action profile $\sigma_{-i}^*(h)$ after every non-terminal history h. Then playing σ_i' guarantees i at least $\underline{v}_i$ in every period, so that his average payoff will be at least $\underline{v}_i$. This would contradict the optimality of σ_i^* in the NE σ^*. □

[38] Even when there are only 2 players, the minimax strategy against P1 might be a mixed strategy of P2. By observing only past actions, P1 does not know if P2 is really randomizing with the correct probabilities.

5 Folk Theorem for Infinitely Repeated Games

5.1 *The folk theorem for infinitely repeated games.* It is natural to ask what payoff profiles can arise in $G^\delta(\infty)$. Write $E\left(G^\delta(\infty)\right)$ for the set of average payoff profiles attainable in SPEs of $G^\delta(\infty)$. Since every SPE is NE, in view of Proposition 94, the most we could hope for are results of the following form: "$\lim_{\delta\to 1} E\left(G^\delta(\infty)\right)$ equals the set of feasible and IR payoffs of G." Theorems along this line are usually called "**folk theorems**", for such results were widely believed and formed part of the economic folklore long before anyone obtained a formal proof.

It is important to remember that folk theorems are not merely efficiency results. They are more correctly characterized as "**anything-goes results**". Not only do they say that there exist SPEs with payoff profiles close to the Pareto frontier, but they also say there exist other SPEs with payoff profiles close to players' minimax payoffs.

The following is a folk theorem for infinitely repeated games with perfect monitoring.

Theorem 95 (Fudenberg and Maskin, 1986)**.** *Write V^* for the set of feasible and strictly IR payoff profiles of G. Assume V^* has full dimensionality. For any $v^* \in V^*$, there exists $\overline{\delta} \in (0,1)$ such that $v^* \in E\left(G^\delta(\infty)\right)$ for all $\delta \in (\overline{\delta}, 1)$.*

Proof. See lecture. □

5.2 *Rewarding minimaxers.* The proof of Theorem 95 is constructive and explicitly defines an SPE with average payoff v^*. To ensure subgame-perfection, the construction must ensure that $-i$ **have an incentive to minimax** i in the event that i deviates. It is possible that the minimax action against i hurts some other player $j \neq i$ so much that j would prefer to be minimaxed instead of minimaxing i. The solution, as we saw in lecture, is to **promise a reward** of $\varepsilon > 0$ in **all future periods** to players who successfully carry out their roles as minimaxers.[39] This way, at a history that calls for players to minimax i, deviating from the minimax action loses an infinite stream of ε payoffs. As players become more patient, this infinite stream of strictly positive payoffs matters far more than the utility cost from finitely many periods of minimaxing i.

[39] The strategy profile used in the proof of Theorem 95 is often called the "stick and carrot strategy". If a player deviates during the normal phase, the deviator is hit with a "stick" for finitely many periods. Then, all the other players are given a "carrot" for having carried out this sanction.

1 Extensions of the Folk Theorem

1.1 *Drop minimaxers' rewards.* Sometimes, no complicated reward schemes as in Theorem 95 are necessary. This is the case when minimaxing i is not particularly costly for her opponents, as the following example demonstrates.

Example 96 (December 2012 Final Exam)**.** Consider an infinitely repeated game with the following symmetric stage game.

	L	C	R
T	$-4,-4$	$12,-8$	$3,1$
M	$-8,12$	$8,8$	$5,0$
B	$1,3$	$0,5$	$4,4$

Construct a **pure** strategy profile of the repeated game with the following properties: (i) the strategy profile is an SPE of the repeated game for all δ close enough to 1; (ii) the average payoffs are $(8,8)$; (iii) in every subgame, both players' payoffs are nonnegative in each period.

Solution:

We quickly verify that each player's pure minimax payoff (i.e., when minimaxers are restricted to using pure strategies) is 1. P1 minimaxes P2 with T, who best responds with R, leading to the payoff profile $(3,1)$. Symmetrically, P2 minimaxes P1 with L, who best responds with B, giving us the payoff profile $(1,3)$. So, $(8,8)$ is feasible and strictly individually rational, even when restricting attention to pure strategies.

However, we cannot directly recite Theorem 95, for the construction there uses a public randomization device in several places – for instance to give the $\varepsilon > 0$ reward to minmiaxers – but the question asks for a pure strategy profile. Even if we are allowed to use public randomizations, we still face the additional restriction that we cannot let any player get a negative payoff in any period, even off-path. If we publicly randomize over some action profiles, then we are restricted to those action profiles in the lower right corner of the payoff matrix in all subgames.

Perhaps the easiest solution is to **build a simpler SPE** and forget about giving the $\varepsilon > 0$ reward to minimaxers altogether. This is possible because for this particular stage game, the minimaxer gets utility 3 while the minimaxee gets utility 1, so it is better to minimax than to get minimaxed. Consider an SPE given by three phases: in **normal phase**, play (M,C); in **minimax P1 phase**, play (B,L); in **minimax P2 phase**, play (T,R). If player i deviates during normal phase, go to minimax Pi phase. If player j deviates during minimax Pi phase, go to minimax Pj phase, where possibly $j = i$. If minimax Pi phase completes without deviations, go to normal phase.

We verify this strategy profile is an SPE for δ near enough 1 using one-shot deviation principle. Due to symmetry, it suffices to verify P1 has no profitable one-shot deviation in any subgame. For any subgame in normal phase, deviating gives at most

$$12 + \delta \cdot 1 + \frac{\delta^2}{1-\delta} \cdot 8, \tag{6}$$

while not deviating gives

$$8 + \delta \cdot 8 + \frac{\delta^2}{1-\delta} \cdot 8. \tag{7}$$

Equation (7) minus equation (6) gives

$$-4 + \delta \cdot 7,$$

which is positive for $\delta \geq \frac{4}{7}$.

For a subgame in the minimax P1 phase, deviating not only hurts P1's current period payoff, but also leads to another period of P1 being minImaxed. So P1 has no profitable one-shot deviation in such subgames for any δ.

For a subgame in the minimax P2 phase, deviating gives at most

$$5 + \delta \cdot 1 + \frac{\delta^2}{1-\delta} \cdot 8, \tag{8}$$

while not deviating gives

$$3 + \delta \cdot 8 + \frac{\delta^2}{1-\delta} \cdot 8. \tag{9}$$

Equation (9) minus equation (8) gives

$$-2 + \delta \cdot 7,$$

which is positive for $\delta \geq \frac{2}{7}$.

Therefore this strategy profile is an SPE whenever $\delta \geq \max\left\{\frac{4}{7}, \frac{2}{7}\right\} = \frac{4}{7}$. ♦

It turns out minimaxer rewards are generally unnecessary when there are only two players[40], as the following theorem shows. In particular, this says we can drop the full-dimensionality assumption from the Fudenberg-Maskin theorem when $n = 2$.

Theorem 97 (Fudenberg and Maskin, 1986)**.** *Write V^* for the set of feasible and strictly IR payoff profiles of G where $n = 2$. For any $v^* \in V^*$, there exists $\overline{\delta} \in (0,1)$ such that $v^* \in E\left(G^\delta(\infty)\right)$ for all $\delta \in (\overline{\delta}, 1)$.*

Proof. We may without loss assume each player's minimax payoff is 0.

Consider a strategy profile with two phases. In the **normal phase**, players publicly randomize over vertices $\{u(a) : a \in A\}$ to get v^* as an expected payoff profile. In the **mutual minimax phase**, P1 plays the minimax strategy against P2 while P2 also plays the minimax strategy against P1, for M periods. If any player deviates in the normal phase, go to the mutual minimax phase. If any player deviates in the mutual minimax phase, restart the mutual minimax phase. If the mutual minimax phase completes without deviations, go to normal phase.

We show that for suitable choice of M, this strategy profile forms an SPE for all δ near 1. Write

$$\overline{v}_i \equiv \max_{a_i, a_{-i} \in A_{-i}} u_i(a_i, a_{-i}).$$

Choose M large enough so that $Mv_i^* \geq 2\overline{v}_i$ for each $i \in \{1,2\}$. Write $\underline{u}_i$ as the payoff to i when i and $-i$ both play the minimax actions against each other. Note that $\underline{u}_i \leq 0$.

Consider a subgame in normal phase. If player i makes a one-shot deviation, she gets at most:

$$\overline{v}_i + \delta\underline{u}_i + \delta^2\underline{u}_i + \cdots + \delta^M\underline{u}_i + \frac{\delta^{M+1}}{1-\delta}v_i^*, \tag{10}$$

while conforming gives

$$v_i^* + \delta v_i^* + \delta^2 v_i^* + \cdots + \delta^M v_i^* + \frac{\delta^{M+1}}{1-\delta}v_i^*. \tag{11}$$

Equation (11) minus equation (10) gives

$$v_i^* - \overline{v}_i + (\delta + \cdots + \delta^M)(v_i^* - \underline{u}_i),$$

which is no less than $-\overline{v}_i + (\delta + \cdots + \delta^M)v_i^*$ since $v_i^* > 0$ and $\underline{u}_i \leq 0$. But for δ close to 1, $\delta + \cdots + \delta^M \geq M/2$, hence implying $-\overline{v}_i + (\delta + \cdots + \delta^M)v_i^* \geq 0$ by the choice of M. So for δ large enough, there are no profitable one-shot deviations in normal phase.

Consider a subgame in the first period of the mutual minimax phase. If player i deviates, she gets at most:

$$0 + \delta\underline{u}_i + \delta^2\underline{u}_i + \cdots + \delta^M\underline{u}_i + \frac{\delta^{M+1}}{1-\delta}v_i^*, \tag{12}$$

where the opponent playing the minimax strategy against i implies that her payoff in the period of deviation is bounded by 0. On the other hand, conforming gives

$$\underline{u}_i + \delta\underline{u}_i + \delta^2\underline{u}_i + \cdots + \delta^M v_i^* + \frac{\delta^{M+1}}{1-\delta}v_i^*. \tag{13}$$

[40] However, the SPE from the proof of Theorem 97 is not allowed in Example 96, as it involves players getting payoffs $(-4,-4)$ in some periods of some off-path subgames.

Equation (13) minus equation (12) gives:

$$(1-\delta^M)\underline{u}_i + \delta^M v_i^*,$$

which is positive for δ sufficiently close to 1 since $\underline{u}_i \leq 0$ and $v_i^* > 0$. This shows i does not have a profitable one-shot deviation in the first period of the mutual minimax phase. *A fortiori*, she cannot have a profitable one-shot deviation in later periods of the mutual minimax phase either.

This completes the proof. □

Example 98 (From old problem sets of Jerry Green)**.** Consider the infinitely repeated game whose stage game is as below and whose common discount factor is $\frac{1}{2}$.

	A	D
A	$2,3$	$1,5$
D	$0,1$	$0,1$

Show that $((A,A),(A,A),\dots)$ cannot be sustained in any SPE path.

Solution:

Consider the incentives of player 2. For this we can use the one-shot deviation principle. On path player 2 is getting 3 each period. The most profitable one-shot deviation of player 2 is to D, and it gives a current gain of $5-3=2$. The heaviest punishment for the deviation is minimaxing player 2 forever after the deviation, which gives a per-period loss of $1-3=-2$. Since the discount factor is $\frac{1}{2}$, the deviation to D in a single period, followed by the punishment of minimaxed forever gives a utility difference of

$$2 + \frac{\frac{1}{2}}{1-\frac{1}{2}} \cdot (-2) = 0.$$

Any weaker punishment (in terms of giving a higher continuation payoff) would cause player 2 to deviate. Thus, to sustain (A,A) on path forever, in the subgame when player 2 deviates player 1 has to play D for all eternity. However, for player 1, A dominates D in the stage game, so playing D forever leads to an average payoff strictly lower than her IR payoff, which is a contradiction! ♦

Example 99 (December 2016 Final Exam)**.** Consider the three player game given by the following two tables (player 3 chooses the matrix).

a_3	a_2	b_2
a_1	$2,2,2$	$1,1,1$
b_1	$1,1,1$	$1,1,1$

b_3	a_2	b_2
a_1	$1,1,1$	$1,1,1$
b_1	$1,1,1$	$2,2,2$

What is the set of payoff triplets that can arise as average equilibrium payoffs of the infinitely repeated game with discount factor δ when δ is close to 1? Justify your answer.

Solution:

Note that the version of the Theorem 95 doesn't help here, because the set of feasible payoffs (the segment connecting $(1,1,1)$ and $(2,2,2)$) is one-dimensional and there are three players.

Since $(2,2,2)$ is NE payoff of the stage game, it is automatically attainable as average SPE payoff, irrespective of the discount factor.

Each player mixing with probabilities $\frac{1}{2}$ between her two strategies is another NE in the stage game, which gives payoff $\left(\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$. Infinite repetition of this Nash profile leads to average SPE payoff $\left(\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$.

Public randomization over the above two SPEs gives us average SPE payoffs the straight line between $\left(\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$ and $(2,2,2)$, both vertices included.

We now show that any payoff strictly below $\frac{5}{4}$ cannot be sustained in any SPE. Denote by

$$\alpha \equiv \inf\{v : (v,v,v) \text{ is SPE payoff}\}$$

the lowest SPE payoff of a player (from symmetry this has to be the same for all players). Take any SPE payoff (v,v,v). Suppose that it is attained when players follow strategy σ. We have seen in the lecture that from all mixture of actions in the first period, there exists a player that can "deviate" and get at least $\frac{5}{4}$. Hence, for σ to be SPE, this player should find such a one-shot deviation not worthwhile. The deviation yields an average payoff at least $(1-\delta)\frac{5}{4}+\delta\alpha$, so $v \geq (1-\delta)\frac{5}{4}+\delta\alpha$. But v is arbitrary, so it follows that $\alpha \geq (1-\delta)\frac{5}{4}+\delta\alpha$, which implies that $\alpha \geq \frac{5}{4}$.

This completes the proof. The set of possible SPE payoffs is the straight line between $\left(\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$ and $(2,2,2)$, both vertices included. ♦

1.2 *The folk theorem for finitely repeated games.* In view of Proposition 90, the stage game G must have multiple NEs for $G(T)$ to admit more than one SPE. Unlike an infinitely repeated game, a finitely repeated game "unravels" because some NE must be played in the last period. However, if G has multiple NEs, then conditioning which NEs get played in the last few periods of $G(T)$ on players' behavior in the early periods of the repeated game provides incentives for cooperation. The following result is not the most general one, but it shows how one can use the multiplicity of NEs in the stage game to incentivize cooperative behavior for most of the T periods.

Proposition 100. *Suppose that each player's stage game payoffs from Nash equilibria can vary. That is, for each $i \in N$, $u_i\left(\overline{\alpha}^{(i)}\right) = \max_{\alpha\in\text{NE}(G)} u_i(\alpha) > \min_{\alpha\in\text{NE}(G)} u_i(\alpha) = u_i\left(\underline{\alpha}^{(i)}\right)$. Write $d_i \equiv \max_{a_i,a_i'\in A_i,a_{-i}\in A_{-i}}\left\{u_i(a_i',a_{-i}) - u_i(a_i,a_{-i})\right\}$ for an upper bound on the deviation utility to player i in the stage game. Let integer M_i be large enough such that $M_i\cdot\left(u_i\left(\overline{\alpha}^{(i)}\right) - u_i\left(\underline{\alpha}^{(i)}\right)\right) \geq d_i$, and let $M \equiv \sum_{i\in N} M_i$. For any feasible payoff profile v^* with $v_i^* \geq u_i\left(\underline{\alpha}^{(i)}\right)$ for each player i, and any integer $T \geq M$, there exists an SPE of $G(T)$ where the average payoff is v^* for all except the last M periods.*

Proof. Consider the following strategy profile. In the first $T-M$ periods, if no one has deviated so far, publicly randomize so that expected payoff profile is v^*. If some players have deviated and player i was the first to deviate, then play $\underline{\alpha}^{(i)}$ for the remainder of these first $T-M$ periods. In the last M periods, if no one deviated in the first $T-M$ periods, then play $\overline{\alpha}^{(1)}$ for M_1 periods, followed by $\overline{\alpha}^{(2)}$ for M_2 periods, ..., finally $\overline{\alpha}^{(n)}$ for M_n periods. If someone deviated in the first $T-M$ periods and i was the first to deviate, then do the same as before except play $\underline{\alpha}^{(i)}$ in the M_i periods where $\overline{\alpha}^{(i)}$ was played.

We use the one-shot deviation principle to argue this strategy profile forms an SPE. At any subgame starting in the first $T-M$ periods without prior deviations, suppose that player i deviates. Compared with conforming to the SPE strategy, player i gains at most d_i in the current period, but gets weakly worse payoffs for the remainder of these first $T-M$ periods as $v_i^* \geq u_i\left(\underline{\alpha}^{(i)}\right)$. In addition, i loses at least d_i utility across M_i periods in the last M periods of the game by choice of M_i. Therefore, player i does not have a profitable one-shot deviation at any subgame starting in the first $T-M$ periods without prior deviations.

At a subgame starting in the first $T-M$ periods with prior deviation, the SPE specifies playing some NE action profile of the stage game thereafter. Deviation can only hurt current period payoff with no effect on the payoffs of any future periods. Similar reasoning holds for subgames starting in the last M periods. □

Example 101 (December 2013 Final Exam)**.** Suppose the following game is repeated T times and each player maximizes the sum of her payoffs in these T plays. Show that, for every $\varepsilon > 0$, we can choose T big enough so that there exists an SPE of the repeated game in which each player's average payoff is within ε of 2.

	A	B	C
A	$2,2$	$-1,3$	$0,0$
B	$3,-1$	$1,1$	$0,0$
C	$0,0$	$0,0$	$0,0$

Solution:

For given $\varepsilon > 0$, choose T large enough such that $\frac{2(T-1)+1}{T} > 2-\varepsilon$. Consider the following strategy profile for both players: in period $t < T$, play A if (A,A) has been played in all previous periods, else play C. In period T, play B if (A,A) has been played in all previous periods, else play C. At a history in period $t \leq T-1$ where (A,A) has been played in all previous periods, a one-shot deviation at most gains 1 in the current period but loses 2 in each of periods $t+1, t+2, \ldots, T-1$, and finally loses 1 in period T. At a history in period $t \leq T-1$ with prior deviation, one-shot deviation hurts current period payoff and does not change future payoffs. At a history in period T, clearly there is no profitable one-shot deviation as this is the last period of the repeated game and the strategy profile prescribes playing a Nash equilibrium of the stage game. ♦

2 Refinements of NE in Extensive and Normal Form Games

2.1 Four refinements. In lecture we studied four refinements of NE for extensive form games: **perfect Bayesian equilibrium (PBE)**, **sequential equilibrium (SE)**, **trembling-hand perfect equilibrium (THPE)**, and **strategically stable equilibrium (SSE)**. Whereas specifying an NE or SPE just requires writing down a profile of strategies, PBE and SE are defined in terms of not only a strategy profile, but also a **belief system** π – that is, a collection of distributions $\pi_j(\cdot|I_j) \in \Delta(I_j)$ over the vertices in information set I_j for each information set of each player j. The four refinements differ in terms of some **consistency conditions** they impose on the belief system.

Definition 102 (Perfect Bayesian equilibrium)**.** A (weak) **perfect Bayesian equilibrium (PBE)** is a strategy profile together with a belief system, (σ, π), so that:

1. For all player $j \in N$ and information set $I_j \in \mathcal{I}_j$, σ_j maximizes expected payoffs starting from information set I_j according to belief system π:
$$u_j(\sigma_j, \sigma_{-j}|I_j, \pi) \geq u_j(\sigma'_j, \sigma_{-j}|I_j, \pi) \text{ for all } \sigma'_j \in \Delta(S_j).$$
2. For all **on-path**[41] information sets I_j, $\pi_j(\cdot|I_j)$ is derived from Bayes' rule.

If an information set I_j is reached with strictly positive probability under σ, then the conditional probability of having reached each vertex $v \in I_j$ given that I_j is reached, $\pi_j(v|I_j)$, is well-defined. On the other hand, we cannot use Bayes' rule to compute the conditional probability of reaching various vertices in an off-path information set, as we would be dividing by 0. As such, PBE places no restrictions on these off-path beliefs.

Definition 103 (Sequential equilibrium)**.** A **sequential equilibrium (SE)** is a strategy profile together with a belief system, (σ, π), so that:

1. For all player $j \in N$ and information set $I_j \in \mathcal{I}_j$, σ_j maximizes expected payoffs starting from information set I_j according to belief system π:
$$u_j(\sigma_j, \sigma_{-j}|I_j, \pi) \geq u_j(\sigma'_j, \sigma_{-j}|I_j, \pi) \text{ for all } \sigma'_j \in \Delta(S_j).$$
2. There exists a sequence of strictly mixed strategies $\left\{\sigma^{(m)}\right\}$ so that $\sigma^{(m)} \to \sigma$, and furthermore $\pi^{(m)} \to \pi$, where for each m, $\pi^{(m)}$ is the unique belief system consistent with $\sigma^{(m)}$ under Bayes' rule.

Though it is not part of the definition, it is easy to show that in an SE, all on-path beliefs are given by Bayes' rule, just as in PBE.

Compared to PBE, SE places some **additional restrictions on off-path beliefs**. Instead of allowing them to be completely arbitrary, SE insists that these off-path beliefs must be attainable as the limiting beliefs of a sequence of strictly mixed strategy profiles that converge to σ – hence the name "sequential equilibrium". Given a strictly mixed $\sigma^{(m)}$, every information set is reached with strictly positive probability. Therefore, the belief system $\pi^{(m)}$ is well-defined, as there exists exactly one such system consistent with $\sigma^{(m)}$ under Bayes' rule.

Importantly, there are **no assumptions of rationality** on the sequence of strategies $\sigma^{(m)}$. It is merely a device used to justify how the belief system π might arise. In particular, there is no requirement that $\sigma^{(m)}$ forms any kind of equilibrium under beliefs $\pi^{(m)}$.

There is one special case where a PBE is automatically an SE.

Proposition 104. *If all non-singleton information sets of all players are on-path in a PBE, then that PBE is an SE.*

Relatedly, there is a case where an extensive form NE is automatically SE.

Proposition 105. *Suppose σ is a strictly mixed Nash equilibrium in an extensive form game. Let π be the unique belief system consistent with σ under Bayes' rule. Then (σ, π) is a sequential equilibrium.*

The next two equilibrium concepts, THPE and SSE, are defined in terms of trembles. A **tremble** $\varepsilon : M \to (0, 1]$ in an extensive form game associates a small, positive probability to each move in each information set, interpreted as the minimum weight that any strategy must assign to the move. That is, the constraint we impose on σ_j for all player $j \in N$, information set $I_j \in \mathcal{I}_j$ and move $m_{I_j} \in M_{I_j}$ is $\sigma_{I_j}(m_{I_j}) \geq \varepsilon(m_{I_j})$.

The strategy profile σ is said to be an **ε-constrained equilibrium** if at each information set I_j, σ_j maximizes j's expected payoff subject to the constraint of minimum weights from the tremble ε. Again, due to strictly mixing, there exists exactly one belief system consistent with σ under Bayes' rule.

[41] An information set is called **on-path** if it is reached with strictly positive probability under σ. Else, it is called **off-path**.

Definition 106 (Trembling-hand perfect equilibrium)**.** A **trembling-hand perfect equilibrium (THPE)** is a strategy profile σ so that **there exists** a sequence of trembles $\left\{\varepsilon^{(m)}\right\}$ converging to 0 and a sequence of strictly mixed strategies $\left\{\sigma^{(m)}\right\}$ so that $\sigma^{(m)} \to \sigma$ and for each m, $\sigma^{(m)}$ is an $\varepsilon^{(m)}$-constrained equilibrium.

Definition 107 (Strategically stable equilibrium)**.** A **strategically stable equilibrium (SSE)** is a strategy profile σ so that **for every** sequence of trembles $\left\{\varepsilon^{(m)}\right\}$ converging to 0, **there exists** a sequence of strictly mixed strategies $\left\{\sigma^{(m)}\right\}$ so that $\sigma^{(m)} \to \sigma$ and for each m, $\sigma^{(m)}$ is an $\varepsilon^{(m)}$-constrained equilibrium.

THPE and SSE are also defined for normal form games, where the tremble ε specifies minimum weights for the different actions of all players.

The following table summarizes some of the key comparisons between these four equilibrium concepts.

	Belief at on-path info. set	Belief at off-path info. set	Robustness to trembles
PBE	Bayes' rule	No restriction	Not robust
SE	Bayes' rule	Limit of beliefs associated with one sequence of strictly mixed profiles	Not robust
THPE	N/A	N/A	Robust to one sequence of trembles
SSE	N/A	N/A	Robust to any sequence of trembles

Finally here are some useful facts:

Fact 108. *For an extensive form game* Γ*, the following inclusions*[42] *hold:*

$$\text{NE}(\Gamma) \supseteq \begin{matrix}\text{PBE}(\Gamma) \\ \text{SPE}(\Gamma)\end{matrix} \supseteq \text{SE}(\Gamma) \supseteq \text{THPE}(\Gamma) \supseteq \text{SSE}(\Gamma).$$

There is no inclusion relationship between PBE(Γ) and SPE(Γ).

Fact 109. *For a finite extensive form game* Γ*,* $\text{THPE}(\Gamma) \neq \varnothing$*, though it is possible that* $\text{SSE}(\Gamma) = \varnothing$*.*

That is, there is always at least one THPE in a finite extensive form game. The immediate implication is that SE, SPE, PBE, and NE are also non-empty equilibrium concepts.

2.2 Some examples. We illustrate these refinement concepts through two examples. The first example shows an extensive form game where we have strict inclusions: $\text{NE}(\Gamma) \supsetneq \text{PBE}(\Gamma) \supsetneq \text{SE}(\Gamma)$.

Example 110 (A modified market entry game)**.** Consider the following modification to the entry game. The entrant (P1) chooses whether to stay out or enter the market. If she enters, nature then determines whether her product is *g*ood or *b*ad, each with 50% probability. Incumbent (P2) observes entry decision, but not whether the product is good or bad. If entrant enters, the incumbent can choose to *A*llow entry, *F*ight, or *F*ight *F*iercely. This extensive game is depicted in Figure 14. Let's write I_2 for P2's information set and abbreviate strategies in the obvious way (eg. (O, F) is the strategy profile where P1 plays *O*ut, P2 plays *F*ight). Restrict attention to pure strategy equilibria.

1. What are the pure strategy **NEs** of this game?

   It is easy to see that (O, F), (O, FF) are NEs. P2's action has no effect on his payoff, since P1 never enters. P1 does not have a profitable deviation either, as playing I yields payoff of -1 if P2 plays F, -9 if P2 plays FF.

   In addition, (I, A) is also an NE. P2 does not have a profitable deviation to F, since doing so yields an expected payoff of $0.5 \cdot (-1) + 0.5 \cdot 2 = 0.75 < 1$. P2 does not have a profitable deviation to FF, since doing so yields an expected payoff of -9.

2. What are the pure strategy **PBEs** of this game?

   The associated strategies must be a subset of NEs.

   There cannot be a PBE with strategy profile (O, FF). No matter what off-path belief $\pi_2(\cdot|I_2)$ P2 holds, he will find it strictly better to play A (which leads to payoff 1) rather than FF (which leads to payoff -9).

[42] Inclusion is in terms of strategy profiles. Technically, NE does not belong to the same universe as PBE and SE, as these later objects require a belief system in addition to a strategy profile.

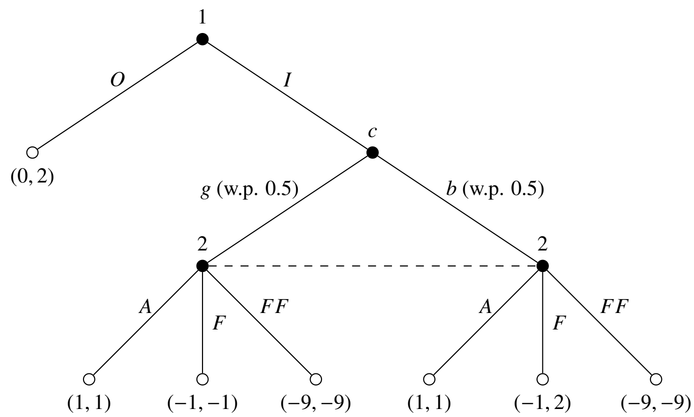

Figure 14: A modified market entry game.

However, for belief $\pi_2(b|I_2) \geq \frac{2}{3}$, $((O, F), \pi_2(\cdot|I_2))$ forms a PBE. Note that I_2 is off-path under strategy profile (O, F), so P2 is allowed to hold any belief. P2's payoff from F is $(1 - \pi_2(b|I_2)) \cdot (-1) + \pi_2(b|I_2) \cdot 2 = 3\pi_2(b|I_2) - 1$. Given belief $\pi_2(b|I_2) \geq \frac{2}{3}$, P2's payoff from F which is greater than his payoff from A or FF.[43]

In addition, $\left((I, A), \pi_2(b|I_2) = \frac{1}{2}\right)$ is another PBE. In fact, this is the only PBE featuring strategy profile (I, A), since the information set I_2 is on-path for this strategy profile and so $\pi_2(\cdot|I_2)$ must be derived from Bayes' rule.

3. What are the pure strategy **SEs** of this game?

   These must form a subset of pure PBEs.

   There is no SE of the form $((O, F), \pi_2(\cdot|I_2))$. This is because, in every strictly mixed behavioral strategy profile $\sigma^{(m)}$, the information set I_2 is reached with strictly positive probability, meaning Bayes' rule requires $\pi_2^{(m)}(b|I_2) = \frac{1}{2}$. But SE requires that $\pi_2^{(m)}(b|I_2) \to \pi_2(b|I_2)$, so we see in any SE of this form we must have $\pi_2(b|I_2) = \frac{1}{2} < \frac{2}{3}$. SE requires that P2's action at the information set maximizes payoff subject to belief, yet under the belief $\pi_2(b|I_2) = \frac{1}{2}$, P2 finds it strictly profitable to deviate to A.

   We finally check that $\left((I, A), \pi_2(b|I_2) = \frac{1}{2}\right)$ is an SE[44]. It is straightforward to verify that actions maximize expected payoff at each information set given belief in $\left((I, A), \pi_2(b|I_2) = \frac{1}{2}\right)$. Now, consider a sequence of strictly mixed strategy profiles $\sigma^{(m)} = \left(\frac{1}{m} O \oplus \left(1 - \frac{1}{m}\right) I, \left(1 - \frac{1}{m}\right) A \oplus \frac{1}{2m} F \oplus \frac{1}{2m} FF\right)$. It is easy to see that $\sigma^{(m)} \to (I, A)$. Furthermore, for each such profile, $\pi_2^{(m)}(b|I_2) = \frac{1}{2}$, so we get $\pi_2^{(m)}(b|I_2) \to \pi_2(b|I_2)$.

The second example illustrates THPE and SSE in a normal form game.

Example 111 (December 2013 Final Exam)**.** In Example 26, we considered the normal form game

	L	R	Y
T	$2, 2$	$-1, 2$	$0, 0$
B	$-1, -1$	$0, 1$	$1, -2$
X	$0, 0$	$-2, 1$	$0, 2$

two pure Nash equilibria, (T, L) and (B, R), as well as infinitely many mixed Nash equilibria, $(T, pL \oplus (1 - p)R)$ for $p \in [\frac{1}{4}, 1)$. Now find all the THPEs and SSEs of this game.

Solution:

[43] Note that this PBE is not an SPE since the strategy profile does not form an NE when restricted to the subgame starting with the chance move. Recall that in lecture we saw an example of an SPE that is not a PBE. This completes the argument that neither the set of SPEs nor the set of PBEs nests the other one.

[44] This does not follow from the non-emptiness of SE as an equilibrium concept, since we have restricted attention to pure equilibria. A game could only have mixed SE.

First we show $(T, pL \oplus (1-p)R)$ is not a THPE for any $p \in [\frac{1}{4}, 1]$ (so we also rule out the pure (T, L)). Suppose there is a sequence of strictly mixed strategy profiles $\sigma^{(m)}$, each of which is an $\varepsilon^{(m)}$-constrained equilibrium for the sequence of trembles, $\varepsilon^{(m)}$, that converge to 0. Since $\sigma_1^{(m)}(B) > 0$ and $\sigma_1^{(m)}(X) > 0$ for each m, R is a **strictly better response** than L against $\sigma_1^{(m)}$ for each m. This means $\sigma_2^{(m)}(L) = \varepsilon^{(m)}(L)$ for each m. Since $\sigma_2^{(m)} \to \sigma_2$, it follows that $\sigma_2(L) = 0 < \frac{1}{4}$. But we know that $\text{THPE}(G) \subseteq \text{NE}(G)$ and $\text{THPE}(G) \neq \varnothing$, so (B, R) must be the unique THPE.

Now we check that (B, R) is also an SSE.[45] Consider **any** sequence of trembles $\{\varepsilon^{(m)}\}$ converging to 0. Suppose P2 plays $\varepsilon_2^{(m)}(L)L \oplus \left(1 - \varepsilon_2^{(m)}(L) - \varepsilon_2^{(m)}(Y)\right) R \oplus \varepsilon_2^{(m)}(Y)Y$. Then P1 gets $2\varepsilon_2^{(m)}(L) - \left(1 - \varepsilon_2^{(m)}(L) - \varepsilon_2^{(m)}(Y)\right)$ from T, $-\varepsilon_2^{(m)}(L) + \varepsilon_2^{(m)}(Y)$ from B, and $-2\left(1 - \varepsilon_2^{(m)}(L) - \varepsilon_2^{(m)}(Y)\right)$ from X. Whenever $\varepsilon_2^{(m)}(L), \varepsilon_2^{(m)}(Y) < 0.1$, it is clear that B is the unique best response, and thus P1 will play $\varepsilon_1^{(m)}(T)T \oplus \left(1 - \varepsilon_1^{(m)}(T) - \varepsilon_1^{(m)}(X)\right) B \oplus \varepsilon_1^{(m)}(X)X$. Similarly, suppose P1 plays $\varepsilon_1^{(m)}(T)T \oplus \left(1 - \varepsilon_1^{(m)}(T) - \varepsilon_1^{(m)}(X)\right) B \oplus \varepsilon_1^{(m)}(X)X$. Then P2 gets $2\varepsilon_1^{(m)}(T) - \left(1 - \varepsilon_1^{(m)}(T) - \varepsilon_1^{(m)}(X)\right)$ from playing L, $2\varepsilon_1^{(m)}(T) + \left(1 - \varepsilon_1^{(m)}(T) - \varepsilon_1^{(m)}(X)\right) + \varepsilon_1^{(m)}(X)$ from playing R, and $-2\left(1 - \varepsilon_1^{(m)}(T) - \varepsilon_1^{(m)}(X)\right) + 2\varepsilon_1^{(m)}(X)$ from playing Y. Whenever $\varepsilon_1^{(m)}(T), \varepsilon_1^{(m)}(X) < 0.1$, it is clear that R is the unique best response, and thus P2 will play $\varepsilon_2^{(m)}(L)L \oplus \left(1 - \varepsilon_2^{(m)}(L) - \varepsilon_2^{(m)}(Y)\right) R \oplus \varepsilon_2^{(m)}(Y)Y$. This means that, given **any** sequence of trembles $\{\varepsilon^{(m)}\}$ converging to 0, eventually in an $\varepsilon^{(m)}$-constrained equilibrium, P1 puts **as much weight as possible** on B while P2 puts as much weight as possible on R – in fact, this happens as soon as the maximum tremble in $\varepsilon^{(m)}$ falls below 0.1. Then, $\sigma^{(m)} \to (B, R)$ shows that (B, R) is an SSE. ♦

3 Signaling Games

3.1 *Strategies, beliefs, and PBEs in signaling games*. Signaling games form an important class of examples within extensive form games with incomplete information. For a schematic representation, see Figure 15.

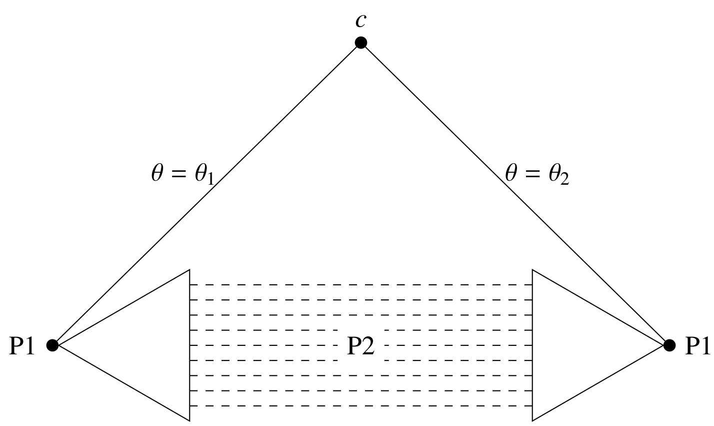

Figure 15: Schematic representation of a signaling game.

Nature determines state of the world, $\theta \in \Theta = \{\theta_1, \theta_2\}$, according to a common prior. P1 is informed of this state. P1 then selects a message from a possibly infinite message set A_1 and sends it to P2. In the buyer-seller example from class, for instance, the state of the world is the quality of the product while the message is a (price, quantity) pair that the seller offers to buyer.

P2 does not observe the state of the world, but observes the message that P1 sends. This means P2 has **one information set for every message** in A_1.

A PBE $(\sigma_1, \sigma_2, \pi_2)$ in the signaling game must then have the following components:

1. $\sigma_1 : \Theta \to \Delta(A_1)$ for P1, that is what to send in each state.
2. $\sigma_2 : A_1 \to \Delta(A_2)$ for P2, that is how to respond to every message that P1 could send (even the off-path messages not sent by P1's strategy).
3. $\pi_2 : A_1 \to \Delta(\Theta)$ for P2, that is what to believe after receiving every message that P1 could send.

[45] Since the set of SSE is not always non-empty, we cannot immediately conclude that (B, R) must be an SSE.

The requirements are that:

1. P2's belief system π_2 is derived from Bayes' rule whenever possible.
2. P2's action after every message a_1 is optimal given $\pi_2(\cdot|a_1)$.
3. P1's action in every state of the world θ is optimal given σ_2.

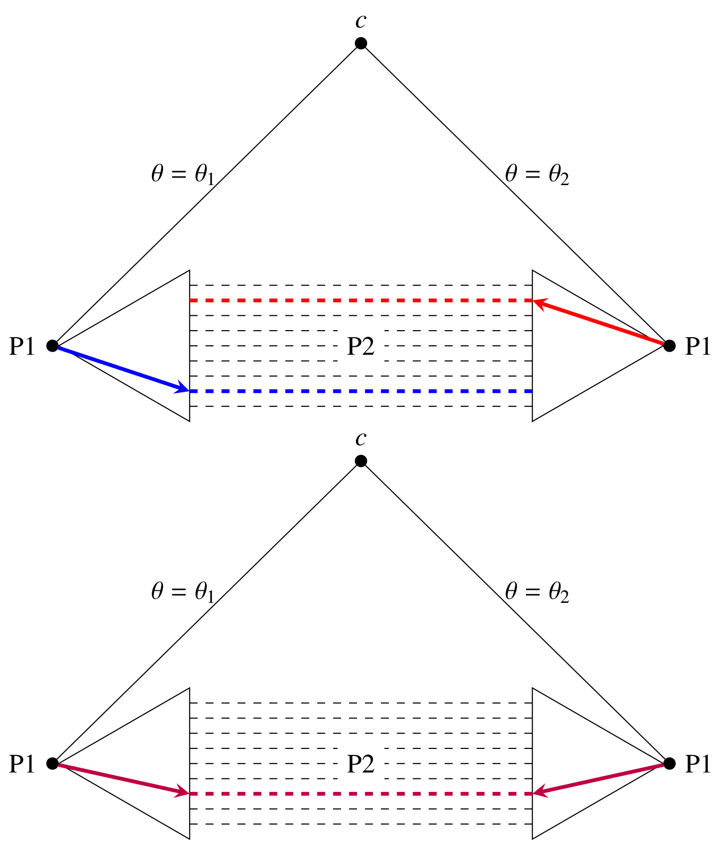

Figure 16: Schematic representations of separating and pooling PBEs.

When there are two states of the world (i.e., two "types" of P1), pure PBEs can be classified into two families, as illustrated in Figure 16. In a **separating PBE**, the two types of P1 send different messages, say $a_1' \neq a_1''$. By Bayes' rule, each of these two messages perfectly reveals the state of the world in the PBE. In a **pooling PBE**, the two types of P1 send the same message, say a'''. By Bayes' rule, P2 should keep his prior about the state of the world after seeing a''' in such a PBE. In a PBE from either family, most of P2's information sets (i.e., messages he could receive from P1) are off-path. PBE allows P2 to hold arbitrary beliefs on these off-path information sets. In fact, we (the analysts) will often want to pick **"pessimistic" off-path beliefs** to help support some strategy profile as a PBE. The following example will illustrate the role of these off-path beliefs in sustaining equilibrium.

3.2 *An example*. We illustrate separating and pooling PBEs in a civil lawsuit example.

Example 112 (Civil lawsuit)**.** Consider a plaintiff (P1) and a defendant (P2) in a civil lawsuit. Plaintiff knows whether she has a strong case (θ_H) or weak case (θ_L), but the defendant does not. Defendant has prior belief that $\pi(\theta_H) = \frac{1}{3}$, $\pi(\theta_L) = \frac{2}{3}$. The plaintiff can ask for a low settlement or a high settlement, $A_1 = \{1, 2\}$. The defendant accepts or refuses, $A_2 = \{y, n\}$. If the defendant accepts a settlement offer of x, the two players **settle out-of-court** with payoffs $(x, -x)$. If defendant refuses, the case goes to trial. If the case is strong ($\theta = \theta_H$), plaintiff wins for sure and the payoffs are $(3, -4)$. If the case is weak ($\theta = \theta_L$), the plaintiff loses for sure and the payoffs are $(-1, 0)$. The extensive form representation of this example is given by Figure 17.

Focus on pure strategy PBEs.

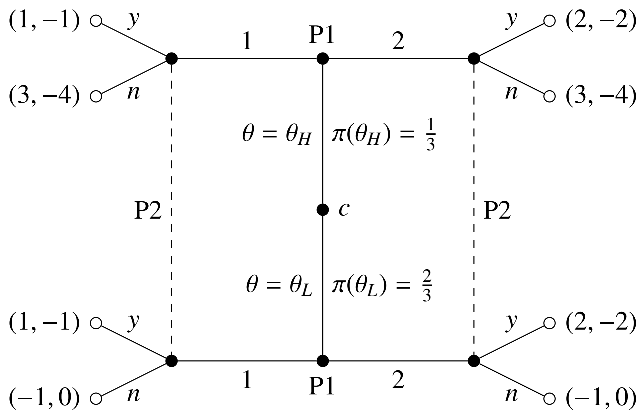

Figure 17: Extensive form representation of the civil lawsuit example.

Separating equilibrium: Typically, there are **multiple** potential separating equilibria, depending on what action each type of P1 plays. Be sure to check all of them.

1. $s_1(\theta_H) = 2$, $s_1(\theta_L) = 1$.

   In any such PBE we must have $\pi_2(\theta_H|2) = 1$, $\pi_2(\theta_L|1) = 1$, $s_2(2) = y$, $s_2(1) = n$, as is illustrated in Figure 18. But this means type θ_L gets -1 in PBE and has a profitable unilateral deviation by playing $s_1'(\theta_L) = 2$ instead. Asking for the high settlement makes P2 think P1 has a strong case, so that P2 will settle and P1 will get 2 instead of -1. Therefore no such PBE exists.

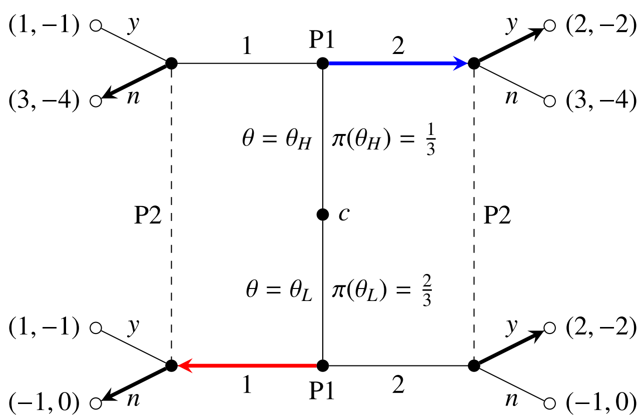

Figure 18: $s_1(\theta_H) = 2$, $s_1(\theta_L) = 1$ is not part of PBE.

2. $s_1(\theta_H) = 1$, $s_1(\theta_L) = 2$.[46]

   In any such PBE we must have $\pi_2(\theta_L|2) = 1$, $\pi_2(\theta_H|1) = 1$, $s_2(2) = n$, $s_2(1) = y$, as is illustrated in Figure 19. But this means type θ_H gets 1 in PBE and has a profitable unilateral deviation by playing $s_1'(\theta_H) = 2$ instead. Asking for the high settlement makes P2 think P1 has a weak case, so that P2 will let the trial go to court. But this is great when P1 has a strong case, giving her a payoff of 3 instead of 1. Therefore no such PBE exists.

[46] It seems very counterintuitive that the plaintiff with a strong case asks for a lower settlement than the plaintiff with a weak case, but this is still a candidate for a separating PBE so we cannot ignore it.

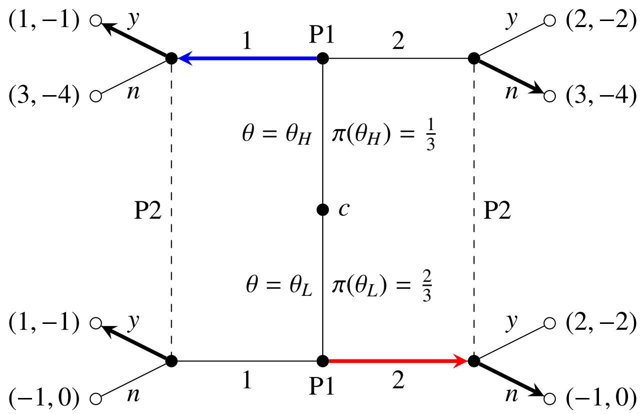

Figure 19: $s_1(\theta_H) = 1$, $s_1(\theta_L) = 2$ is not part of PBE.

Pooling equilibrium: In a pooling equilibrium all types of P1 play the same action. When this "pooled" action a_1^* is observed, P2's posterior belief is the **same as the prior**, $\pi_2(\theta|a_1^*) = \pi(\theta)$, since the action carries no additional information about P1's type. When any other action is observed (i.e. an off-path action is observed), PBE allows P2's belief to be **arbitrary**. Every member of A_1 could serve as a pooled action, so we need to **check for all of them systematically**.

1. $s_1(\theta_H) = s_1(\theta_L) = 1$.

   In any such PBE we must have $\pi_2(\theta_H|1) = \frac{1}{3}$. Under this belief, P2's expected payoff to $a_2 = n$ is $\frac{1}{3} \cdot (-4) + \frac{2}{3} \cdot 0 = -\frac{4}{3}$, while playing $a_2 = y$ always yields -1. Therefore in any such PBE we must have $s_2(1) = y$, which gives both types of P1 payoff 1, as is illustrated in Figure 20. But then the θ_H type of P1 has a profitable unilateral deviation of $s_1'(\theta_H) = 2$, regardless of what $s_2(2)$ is! If $s_2(2) = y$, that is P2 accepts the high settlement, then type θ_H P1's deviation gives her a payoff of 2 rather than 1. If $s_2(2) = n$, that is P2 refuses the high settlement, then this is even better for the type θ_H P1 as she will get a payoff of 3 when the case goes to court. Therefore no such PBE exists.

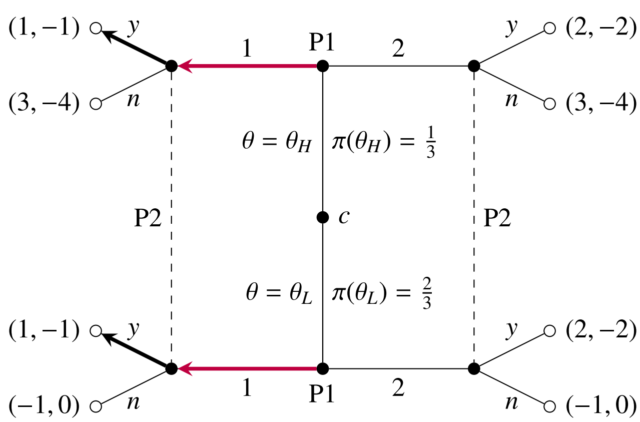

Figure 20: $s_1(\theta_H) = s_1(\theta_L) = 1$ is not part of PBE.

2. $s_1(\theta_H) = s_1(\theta_L) = 2$.

   In any such PBE we must have $\pi_2(\theta_H|2) = \frac{1}{3}$. Under this belief, P2's expected payoff to $a_2 = n$ is $\frac{1}{3} \cdot (-4) + \frac{2}{3} \cdot 0 = -\frac{4}{3}$, while playing $a_2 = y$ always yields -2. Therefore in any such PBE we must have $s_2(2) = n$, which give type θ_H payoff of 3 and type θ_L payoff of -1. Type θ_H does not have incentive to deviate as she already gets her maximum payoff. In order to prevent a deviation by type θ_L, we must ensure $s_2(1) = n$ as well, as is illustrated in Figure 21. Else, if P2 accepts the low settlement offer, θ_L would have a profitable deviation: offering the low settlement instead of following the pooling action of high settlement yields her a payoff of 1 instead of -1. Whether $s_2(1) = n$ is optimal for P2 **depends on the belief**, $\pi_2(\theta_H|1)$. This is an **off-path belief** and PBE allows such beliefs to be arbitrary. Suppose $\pi_2(\theta_H|1) = \lambda \in [0, 1]$. Then P2's expected payoff to playing $s_2(1) = n$ is

$\lambda \cdot (-4) + (1-\lambda) \cdot 0 = -4\lambda$, while $s_2(1) = y$ yields -1 for sure. Therefore to ensure P2 $s_2(1) = n$ is optimal given belief, we need $\lambda \leq \frac{1}{4}$. If P2's off-path belief is that P1 has a strong case with probability less than $\frac{1}{4}$ upon seeing a low-settlement offer, then it is optimal for P2 to reject such low-settlement offers and θ_L will not have a profitable deviation. In summary, there is a family of pooling equilibria where $s_1(\theta_H) = s_1(\theta_L) = 2$, $s_2(1) = s_2(2) = n$, $\pi_2(\theta_H|2) = \frac{1}{3}$, $\pi_2(\theta_H|1) = \lambda$ where $\lambda \in [0, \frac{1}{4}]$. Crucially, it is the judicious choice of **off-path belief** $\pi_2(\cdot|1)$ that sustains an action of $s_2(1) = n$, which in turn sustains the pooling equilibrium.

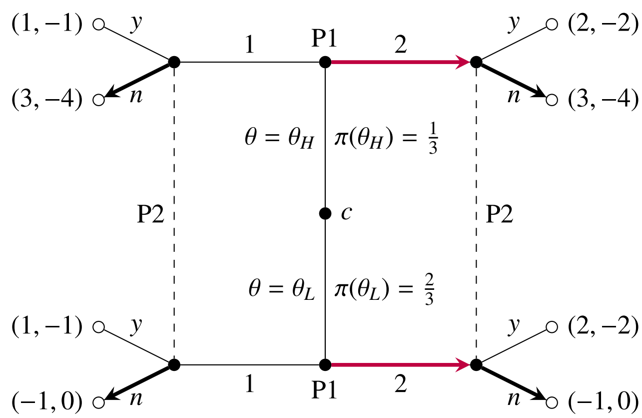

Figure 21: $s_1(\theta_H) = s_1(\theta_L) = 2$ can be part of PBE.

To sum up, any pure strategy PBE in this game is a pooling equilibrium (s, π) with $s_1(\theta_H) = s_1(\theta_L) = 2$, $s_2(1) = s_2(2) = n$, $\pi_2(\theta_H|2) = \frac{1}{3}$, $\pi_2(\theta_H|1) = \lambda$ where $\lambda \in [0, \frac{1}{4}]$. ♦

3.3 Intuitive criterion. Some of the PBEs seem fragile, and can be broken using a speech as we saw in the lecture. This is a heuristic, since it is hard to explicitly model speech. Cho and Kreps (1987) formalize this idea and introduce the **intuitive criterion**. It aims to reduce possible outcome scenarios by: (i) restricting the type space to types of agents who could obtain higher utility levels by deviating to off-path messages, and (ii) by considering in this subset of types the types for which the off-path message is not dominated under opponent's best response.

Definition 113 (Intuitive criterion)**.** A PBE (s_1, s_2, π_2) in a signaling game satisfies the **intuitive criterion** if there do not exist $(\hat{a}_1, \hat{a}_2, \hat{\theta})$ such that:

1. $\hat{a}_1 \notin \{s_1(\theta_1), s_1(\theta_2)\}$.
2. $u_1(\hat{a}_1, \hat{a}_2, \hat{\theta}) > u_1(s_1(\hat{\theta}), s_2(s_1(\hat{\theta})), \hat{\theta})$.
3. $u_1(\hat{a}_1, a_2, \theta) < u_1(s_1(\theta), s_2(s_1(\theta)), \theta)$ for all $a_2 \in A_2$ and $\theta \neq \hat{\theta}$.
4. $\hat{a}_2 \in \arg\max_{a_2 \in A_2} u_2(\hat{a}_1, a_2, \hat{\theta})$.

Cho and Kreps (1987): "Despite the name we have given it, the intuitive criterion is not completely intuitive." P2 is trying to infer P1's type based on the off-path message $\hat{a}_1$. The intuitive criterion makes the following restriction: If for type θ, every response P2 might make after $\hat{a}_1$ yields strictly less payoff than equilibrium, then P2 "should be sure" that type θ would not deviate to $\hat{a}_1$. Why not restrict the off-path beliefs directly? One answer is that it causes existence problems in games in which P1 has actions that are dominated for all types.

In the civil lawsuit example above, all of the pooling PBEs satisfy the intuitive criterion. The only off-path message $\hat{a}_1$ in the pooling PBE is the low settlement offer, $\hat{a}_1 = 1$.

If $\hat{\theta} = \theta_L$, then by condition 4, $\hat{a}_2 = n$. But this shows that condition 2 must not hold, since θ_L gets the same payoff in the PBE as under $(\hat{a}_1, \hat{a}_2, \hat{\theta})$ – the defendant rejects the settlement in both cases.

If $\hat{\theta} = \theta_H$, then by condition 4, $\hat{a}_2 = y$. But this shows that condition 2 must not hold, since θ_H was actually getting higher payoff in PBE when defendant rejects settlement than under $(\hat{a}_1, \hat{a}_2, \hat{\theta})$, where defendant accepts settlement.

So there are no $(\hat{a}_1, \hat{a}_2, \hat{\theta})$ satisfying conditions 1 through 4, meaning the high settlement pooling PBEs satisfy the intuitive criterion.

4 The End

"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."

— Alice in Wonderland, on how to survive grad school at Harvard

7 References

Aumann, Robert J., Sergiu Hart, and Motty Perry, (1997). "The Absent-Minded Driver." *Games and Economic Behavior* 20 (1):102–116.

Brandenburger, Adam and Eddie Dekel, (1993). "Hierarchies of Beliefs and Common Knowledge." *Journal of Economic Theory* 59 (1):189–198.

Cho, In-Koo and David M. Kreps, (1987). "Signaling Games and Stable Equilibria." *The Quarterly Journal of Economics* 102 (2):179–221.

Fudenberg, Drew, David Levine, and Eric Maskin, (1994). "The Folk Theorem with Imperfect Public Information." *Econometrica* 62 (5):997–1039.

Fudenberg, Drew and Eric Maskin, (1986). "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information." *Econometrica* 54 (3):533–554.

Harsanyi, John C., (1967). "Games with Incomplete Information Played by Bayesian Players, I-III Part I. The Basic Model." *Management Science* 14 (3):159–182.

———, (1973). "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points." *International Journal of Game Theory* 2 (1):1–23.

Kandori, Michihiro and Hitoshi Matsushima, (1998). "Private Observation, Communication and Collusion." *Econometrica* 66 (3):627–652.

Keynes, John Maynard, (1936). *The General Theory of Employment, Interest and Money*. Macmillan.

Kuhn, Harold William, (1953). "Extensive Games and the Problem of Information." In *Contributions to the Theory of Games*, vol. 2. Princeton University Press, 193–216.

Ledoux, Alain, (1981). "Concours résultats complets." *Les victimes se sont plu á jouer le* 14:10–11.

Mas-Collell, Andreu, Michael D. Whinston, and Jerry R. Green, (1995). *Microeconomic Theory*. Oxford University Press.

Maschler, Michael, Eilon Solan, and Shmuel Zamir, (2013). *Game Theory*. Cambridge: Cambridge University Press.

Maskin, Eric, (1999). "Nash Equilibrium and Welfare Optimality." *The Review of Economic Studies* 66 (1):23–38.

Mertens, Jean-François and Shmuel Zamir, (1985). "Formulation of Bayesian analysis for games with incomplete information." *International Journal of Game Theory* 14 (1):1–29.

Myerson, Roger B., (1981). "Optimal Auction Design." *Mathematics of Operations Research* 6 (1):58–73.

———, (2013). *Game Theory*. Harvard University Press.

Nash, John, (1950). "Equilibrium Points in N-Person Games." *Proceedings of the National Academy of Sciences* 36 (1):48–49.

———, (1951). "Non-Cooperative Games." *Annals of Mathematics* 54 (2):286–295.

Osborne, Martin J. and Ariel Rubinstein, (1994). *A Course in Game Theory*. Cambridge: The MIT Press.

Sugaya, Takuo, (Forthcoming). "Folk Theorem in Repeated Games with Private Monitoring." *The Review of Economic Studies* .

von Neumann, John, (1928). "Zur theorie der gesellschaftsspiele." *Mathematische Annalen* 100 (1):295–320.